Department of Mathematics and Systems Analysis

# Integration in a Normal World: Fractional Brownian Motion and Beyond

Lauri Viitasaari

Aalto University

DOCTORAL DISSERTATIONS

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# Integration in a Normal World: Fractional Brownian Motion and Beyond

**Lauri Viitasaari**

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# Abstract

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**Abstract**

This thesis is about stochastic integration with respect to Gaussian processes that are not semimartingales. Firstly, we study approximations of integrals with respect to fractional Brownian motion and derive an upper bound for an average approximation error. Secondly, we study the existence of pathwise integrals with respect to a wide class of Gaussian processes and integrands. We prove the existence of two different notions of pathwise integrals. Moreover, these two different integrals coincide. As an application of these results, the thesis contains integral representations for arbitrary random variables. Finally, we study a certain model involving a Gaussian process and provide estimators for different parameters. We apply Malliavin calculus and divergence integrals to obtain central limit theorems for our estimators.

**Keywords** Gaussian process, Fractional Brownian motion, Approximation error, Pathwise integrals, Integral representation, Parameter estimation, Divergence integrals

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**Tiivistelmä**

Tämä väitöskirja käsittelee stokastista integrointia Gaussisten prosessien suhteen, jotka eivät ole semimartingaaleja. Aluksi työssä tutkitaan approksimaatioita integraaleille fraktionaalisen Brownin liikkeen suhteen ja johdetaan yläraja keskimääräiselle approksimaatiovirheelle. Seuraavaksi työssä tutkitaan poluttaisten integraalien olemassaoloa laajalle joukolle Gaussisia prosesseja ja integrandeja. Työssä todistetaan kahden erilaisen poluttaisen integraalin olemassaolo. Lisäksi työssä näytetään, että nämä kaksi erilaista integraalia yhtyvät. Sovelluksena näistä tuloksista väitöskirjassa johdetaan integraaliesitys mielivaltaiselle satunnaismuuttujalle. Lopuksi työssä tutkitaan erästä Gaussisen prosessin sisältävää mallia ja määritellään estimaattorit mallin eri parametreille. Työssä johdetaan keskeiset raja-arvolauseet määritellyille estimaattoreille hyödyntäen Malliavin laskentaa ja divergenssi-integraaleja.

**Avainsanat** Gaussinen prosessi, fraktionaalinen Brownin liike, approksimaatiovirhe, poluttainen integraali, integraaliesitys, parametrin estimointi, divergenssi-integraali

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# Preface

I started my studies on stochastics with prof. Esko Valkeila as my mentor by making a Bachelor's thesis on random walks during summer in the year of 2007. Today, almost six and a half years later, I find myself still on the same road except now I am finishing a third thesis with Esko as my mentor.

Firstly, and mostly, I express my gratitude to the late Esko Valkeila, my former advisor, supervisor, mentor and a friend, for all his patience, help, support, and kindness during these years.

I am also very grateful to prof. Olavi Nevanlinna who has taken care of me after our journey together with Esko ended with unfortunate health issues. Olavi has been supportive and helpful with all the problems I have had. For financial support my thanks goes to prof. Paavo Salminen and the Finnish Doctoral Programme in Stochastics and Statistics.

I am deeply grateful to my current advisor prof. Tommi Sottinen for all his valuable comments as well as his support during the troublesome times. He has been guiding me in the right direction.

I also give special thanks to Dr. Ehsan Azmoodeh for his guidance and collaboration during this project. He has had a tremendous influence on the research presented in this thesis.

I am also grateful to prof. David Nualart, my opponent, for coming to Helsinki to attend the defense in the middle of his many duties, and to the pre-examiners, prof. Paavo Salminen and prof. Yuliya Mishura, for their valuable comments on my thesis.

I thank all my fellow stochasticians who have helped me during my studies for their support and friendship. Especially I wish to thank Ari-Pekka Perkkiö and Lasse Leskelä for all the important mathematical evenings we have shared.

Warm smiles to the whole department of Mathematics and Systems analysis at Aalto University School of Science, you have all been very friendly. Special thanks go to the members of the coffee room gang. It is more than important

to have some small breaks together with friends with whom one can share ideas as well as have civilized discussions on different important topics such as politics, sports, and other non-mathematical matters. I trust that with these discussions we have made the world a better place.

I would also like to name few of my closest friends who have influenced this project in one way or another. Atte Aalto, Samuli Leppänen, Helle Majander, Olli Myyrä, and Tatu Viitasaari, I give you many thanks. Especially I thank Olli for correcting my English language.

Finally, I want to thank my family and all my other friends for their support and friendship. Especially, my sincerest gratitude belongs to my beloved wife Ninnu, you have been very patient and loving. Moreover, I express my love to my wonderful daughters Melina, Nea, and Vella. On a daily basis you remind me that there exist other things besides mathematics.

Espoo, January 17, 2014,

Lauri Viitasaari

# Contents

# List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

**I** L. Viitasaari and E. Azmoodeh. Rate of Convergence for Discretization of Integrals with Respect to Fractional Brownian Motion. *Journal of Theoretical Probability*, DOI: 10.1007/s10959-013-0495-y, 27 pages, May 2013.

**II** L. Viitasaari and T. Sottinen. Pathwise Integrals and Itô–Tanaka Formula for Gaussian Processes. *http://arxiv.org/abs/1307.3578*, 24 pages, January 2014.

**III** L. Viitasaari. Integral Representation of Random Variables with Respect to Gaussian Processes. Accepted for publication in *Journal of Theoretical Probability*, http://arxiv.org/abs/1307.7559, 17 pages, January 2014.

**IV** L. Viitasaari and E. Azmoodeh. Parameter Estimation Based on Discrete Observations of Fractional Ornstein-Uhlenbeck Process of the Second Kind. *http://arxiv.org/abs/1304.2466*, 23 pages, January 2014.

# Author's Contribution

### Publication I: "Rate of Convergence for Discretization of Integrals with Respect to Fractional Brownian Motion"

This article is a joint discussion with Ehsan Azmoodeh and the problem is addressed by Ehsan Azmoodeh. Most of the results and proofs are a joint work with Azmoodeh. The exception is the main crossing lemma presented in the Appendix which is discovered by the author. The author is responsible for most of the writing.

### Publication II: "Pathwise Integrals and Itô–Tanaka Formula for Gaussian Processes"

This article is a joint discussion with Tommi Sottinen. The problem originates from the author. Moreover, most of the main results with proofs together with the main lemma in the appendix present an independent study of the author. The author is responsible for most of the writing.

### Publication III: "Integral Representation of Random Variables with Respect to Gaussian Processes"

This is an individual work of the author.

## Publication IV: "Parameter Estimation Based on Discrete Observations of Fractional Ornstein-Uhlenbeck Process of the Second Kind"

This is a joint discussion with Ehsan Azmoodeh and the problem is addressed by Ehsan Azmoodeh. All of the results are a joint work with Azmoodeh except the technical computations, presented in the appendix, which are individual work of the author. The author is responsible for most of the writing.

# 1. Introduction

Gaussian processes are an important class of stochastic processes. On one hand, in their own way they are simple to analyse and there exist many interesting results for Gaussian processes which make them easier to handle. On the other hand, Gaussian distribution fits well to many applications or at least, thanks to the central limit theorem, one can approximate the situation under study with Gaussian distribution. Moreover, different stylized facts can be added to the model by manipulating the covariance structure. For these two reasons Gaussian processes are widely applied in different areas. To simply name a few, many phenomenons in physics, chemistry, biology, statistics, queuing theory, machine learning, economy, or sociology can be modelled with Gaussian processes.

Similarly, in many applications it is of interest to study integrals with respect to some stochastic processes. For instance in financial mathematics, a stochastic process $X$ is viewed as the driving process of the stock price and the integral of $\psi$ with respect to $X$ can be viewed as the value of a portfolio, where $\psi_t$ represents the amount of stocks on the portfolio at time $t$. However, usually the stochastic processes under study are not differentiable and consequently, standard integration cannot be applied.

### Early history of stochastic integration

The early history of stochastic integration (for longer introduction to the history of stochastic integration and mathematical finance, see [28] which is our main reference here) can be considered to be originated with standard Brownian motion which is perhaps the most studied Gaussian process. The name for Brownian motion is due to botanist Robert Brown who noted in his studies back in 1827 that particles in water seemed to move through the water. Despite these studies it wasn't until around the beginning of the 20th century that the first attempts were made to model Brownian motion mathematically. These attempts are traced back to three independent sources: Thiele [60] in his studies on time-series, Bachelier [5, 6] (who is currently seen as the founder of modern Mathematical finance) in his study of the Paris stock market, and Einstein [17] who was also modelling the motion of small particles in liquid.

The next step towards modern stochastic integration theory was taken by Wiener on his construction for Brownian motion in 1923, and as a result of his contributing study, Brownian motion is also called Wiener process. Contributing groundwork for stochastic integration theory was also done by Kolmogorov [31] whose study was motivated by Markov processes. Before turning to Itô who is seen as the father of stochastic integration theory we also wish to mention Vincent Doeblin (see [10]). Doeblin had many great modern ideas and perhaps he could have contributed in a significant way to the development of the theory. However, Doeblin was drafted during World War II and he volunteered to go to the front lines. Before he joined the lines he put sketches of his ideas into a safe box which was only to be opened by himself or after 100 years had passed. Unfortunately, the battle went ill and Doeblin burned his notes and took his own life. Consequently, the safe box was opened as late as May 2000 by the request of Doeblin's brother.

Finally, the most important step in the development of the field was the contribution of Kyoshi Itô (for a summary on Itô's work, see Varadhan and Stroock [61]) and nowadays stochastic integrals with respect to semimartingales are referred to as Itô integrals. Later on (and meanwhile) other persons have also contributed to the field and nowadays, thanks to Doob, Hunt, Meyer, Watanabe, and McKean among others, stochastic integration with respect to semimartingales is well-developed.

## "State of the Art"

Recently the development in both Gaussian processes and stochastic integration has been on generalisations to cover other processes than semimartingales or in particular, standard Brownian motion. For instance, while Brownian motion has stationary and independent increments, empirical studies in many fields of applications show that the assumption of independent increments is not always very fitting.

Perhaps the most simple process that can capture this phenomenon is fractional Brownian motion; a generalisation of standard Brownian motion. Fractional Brownian motion is still Gaussian and has stationary increments, but unlike standard Brownian motion, the increments of fractional Brownian motion are dependent. More precisely, fractional Brownian motion depends on a parameter, the so-called Hurst index, $H \in (0,1)$ and the value of $H$ determines the dependence structure. In particular, the case $H = \frac{1}{2}$ corresponds with standard Brownian motion.

Fractional Brownian motion was actually already studied in 1940 by Kolmogorov in the context of modelling turbulence [32, 33]. Fractional Brownian motion appears also on works by Hunt [24], Lévy [36], Yaglom [62], Lamperti [35] and Molchan and Golosov [45] although Lévy studied slightly different process. However, the name fractional Brownian motion comes from the paper by Mandelbrot and Van Ness [41] although the name "Hurst-index" for the parameter $H$ is due to British hydrologist H. Hurst [25].

While fractional Brownian motion is an interesting process for many applications, there is one serious problem: fractional Brownian motion is not a semimartingale except in the case of standard Brownian motion. Consequently, integration with respect to fractional Brownian motion is an interesting subject as classical semimartingale techniques cannot be applied.

In order to define stochastic integration with respect to Gaussian processes that are not semimartingales, there are two main approaches. The first one is pathwise integration techniques which are like "$\omega$-by-$\omega$"-integration techniques, and the second one is Skorokhod integrals or divergence integrals. In this thesis the main emphasis is on pathwise techniques on which we give some more details.

For pathwise stochastic integration techniques there are several different approaches, and we wish to mention three of them which are used in this thesis. Firstly, we would like to mention integration based on $p$-variations developed by Young [63] and integration based on Hölder continuity of the corresponding processes developed by Zähle [64]. Namely, if the integrand and the integrator are together smooth enough, then the corresponding integral exists as a limit of Riemann-Stieltjes sums. The second approach is a forward integral introduced by Hans Föllmer [19] and this approach is often referred to as Föllmer integral. The Föllmer integral is a natural way to define integrals in many applications as it is defined as a limit of certain type Riemann-Stieltjes sums. Finally, the third approach we apply in this thesis is the so-called generalised Lebesgue-Stieltjes integral. In the context of stochastic processes, this type of integration in fractional Besov-type spaces was introduced by Nualart and Răşcanu [50].

When considering stochastic integration, all of the mentioned integration techniques have some drawbacks. Firstly, integration based on $p$-variations or Hölder continuity of corresponding processes have a natural interpretation as the integral exists as the limit of corresponding Riemann-Stieltjes sums. However, for this approach the class of integrands is significantly limited as even the simple process of form $\mathbf{1}_{\{X_t>a\}}$ usually has unbounded $p$-variations. Consequently, Young-integration techniques cannot be applied. Similarly, while the definition of Föllmer integral usually has a natural interpretation, the existence of the integral can sometimes be a difficult question. For instance, in the original paper Föllmer proved the existence of the integral by proving that such an integral satisfies certain Itô formula, and the existence of other terms implies the existence of the stochastic integral. Similarly in many other publications in the literature, the existence of Föllmer integral is usually not proved directly. Finally, for generalised Lebesgue-Stieltjes integrals the existence of the integral is usually, at least in principle, a solvable problem. However, looking at the definition of the integral it is not obvious what a satisfactory interpretation in view of applications would be. Furthermore, the same problem occurs if one considers divergence integrals. For instance in financial mathematics, many interesting results such as integral representations or hedging equations in finance can be derived using divergence integral. However, economical interpretation of the hedging equation is difficult [9, 59].

## On this thesis

The main emphasis in the literature has been in divergence integrals together with many applications. Furthermore, for pathwise integrals one usually assumes that the integrands have enough path regularity (such as Hölder continuity or bounded $p$-variation) or that the integrator is of a certain type. For instance, fractional Brownian motion as integrator has received a lot of attention.

This thesis contributes to the field in several ways. Firstly, we study the existence of the mentioned integrals for integrands which are not Hölder continuous nor of bounded $p$-variation. In particular, we cover the indicators $\mathbf{1}_{\{X_u>a\}}$. As an application, we find integral representation for arbitrary random variables as pathwise integrals with respect to Gaussian processes. In this sense we extend similar results derived for fractional Brownian motion to a much wider class of Gaussian processes. In particular, the Gaussian processes under study are not semimartingales. On the other hand, all processes under study are Hölder continuous of order $\alpha > \frac{1}{2}$. In particular, fractional

Brownian motion with Hurst index $H > \frac{1}{2}$ belongs to the class of processes under study.

Secondly, we study the connection between Föllmer integral and generalised Lebesgue–Stieltjes integral. More precisely, the results that are derived for fractional Brownian motion and what we extend to more general Gaussian processes, apply machinery developed for generalised Lebesgue-Stieltjes integrals. On the other hand, we wish to interpret the integral as a limit of Riemann-Stieltjes sums i.e. as a Föllmer integral. In all of our results, the integral can be understood as a generalised Lebesgue-Stieltjes integral or as a Föllmer integral. Moreover, the integrals coincide. In details, our research in Publication I considers the rate of convergence of Riemann-sums to the corresponding Föllmer integral where a derivative of convex function is integrated with respect to fractional Brownian motion. Furthermore, in Publication II we prove the existence of Föllmer integrals of a certain type for a wide class of Gaussian processes which include the particular case of fractional Brownian motion. Similarly, in Publication III we apply the results of Publication II to define integral representations for arbitrary processes with respect to Gaussian processes. Finally, the tools provided by the theory of Malliavin calculus and divergence integrals are applied in Publication IV, where we consider parameter estimation in a certain fractional Ornstein-Uhlenbeck model.

This thesis consists of two parts. The first part is a short introduction to the topics we consider in this thesis. Mostly we only list the properties and well-known results on different topics. For some important results we also present proofs or at least the key points of the proofs. These proofs are gathered from the literature, and for the derivation the author has no contribution what so ever. The second part contains the articles themselves.

# 2. Gaussian processes

This thesis is about Gaussian processes. In this chapter we introduce the basic facts of Gaussian processes used in the articles. Especially, we list the basic properties of fractional Brownian motion. For more details on Gaussian processes, we refer to books by Adler [1], Hida and Hitsuda [23], Ibragimov and Rozanov [26], Lifshits [38] or Marcus and Rosen [42]. For recently published books, see another book by Lifshits [39].

## 2.1 General facts

**Definition 2.1.1.** *A Gaussian process $X = (X_t)_{t\geq 0}$ is a stochastic process such that for any finite collection of time points $t_1, \ldots, t_n \geq 0$ the random vector $(X_{t_1}, \ldots, X_{t_n})$ is a multivariate Gaussian random variable.*

**Definition 2.1.2.** *Let $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ be two Gaussian processes. We denote $X \overset{law}{=} Y$ if the processes have same finite dimensional distributions i.e. for any time points $t_1, \ldots, t_n \geq 0$ the random vectors $(X_{t_1}, \ldots, X_{t_n})$ and $(Y_{t_1}, \ldots, Y_{t_n})$ have the same multivariate distribution.*

In what follows we assume that the process is centred i.e. $\mathbb{E}[X_t] = 0$ for every $t \geq 0$.

**Definition 2.1.3.** *A covariance function of a centred stochastic process $X = (X_t)_{t\geq 0}$ is a function $R : \mathbb{R}_+^2 \to \mathbb{R}$ defined by*

$$R(s,t) = \mathbb{E}[X_t X_s].$$

**Remark 2.1.1.** *Any covariance function $R$ is non-negative definite i.e. for any $t_1, \ldots, t_n \geq 0$ and $z_1, \ldots, z_n \in \mathbb{R}$ we have*

$$\sum_{j=1}^{n} \sum_{k=1}^{n} R(t_j, t_k) z_j z_k \geq 0.$$

Evidently, for any Gaussian process there is a unique non-negative definite covariance function $R(s,t)$. On the other hand, it is known that for Gaussian processes the law is uniquely determined by the mean and the covariance function in a sense that for any two Gaussian processes $X$ and $Y$ with the same mean and covariance function we have $X \overset{\text{law}}{=} Y$. Consequently, for any non-negative definite function $R : \mathbb{R}^2 \to \mathbb{R}$ there is a unique (in law) centred Gaussian process $X$ with covariance structure $R(s,t)$.

We now introduce two important classes of processes which play a large role in applications. Gaussian examples are given in section 2.3.

**Definition 2.1.4.** *A process $X = (X_t)_{t\geq 0}$ is stationary if for every $h, t \geq 0$ we have*

$$X_{t+h} \overset{law}{=} X_t.$$

Consequently, the covariance function of a stationary Gaussian process satisfies

$$R(s,t) = R(0, |t-s|) =: r(|t-s|).$$

**Definition 2.1.5.** *A process $X = (X_t)_{t\geq 0}$ has stationary increments if for every $h, t \geq 0$ we have*

$$X_{t+h} - X_t \overset{law}{=} X_h - X_0.$$

For stationary sequences the following property is important for many applications.

**Definition 2.1.6.** *A stationary sequence $(\xi_n)_{n\geq 0}$ of random variables exhibits long-range dependence if the autocorrelation function $r(k)$ satisfies*

$$\sum_{k=0}^{\infty} r(k) = \infty.$$

*If $\sum_{k=0}^{\infty} r(k) < \infty$, then the sequence $(\xi_n)_{n\geq 0}$ exhibits short-range dependence.*

**Remark 2.1.2.** *The definition for long-range dependence differs in the literature. For details see Beran [7] or Giraitis et al. [22].*

**Definition 2.1.7.** *Let $H > 0$. A process $X = (X_t)_{t\geq 0}$ is $H$-self-similar if for every $a > 0$*

$$(X_{at})_{t\geq 0} \overset{law}{=} (a^H X_t)_{t\geq 0}.$$

Note that if $X$ is $H$-self-similar, it follows that

$$Var(X_t) = t^{2H} Var(X_1)$$

provided that $X$ is square-integrable.

We give one more result related to Gaussian processes (taken from [42]).

**Theorem 2.1.1.** *A Gaussian process $X$ with covariance function $R$ is Markovian if and only if*

$$R(s,u) = \frac{R(s,t)R(t,u)}{R(t,t)}$$

*for every $s \leq t \leq u$.*

### 2.1.1 Path properties

In this thesis we mainly consider processes on compact interval $[0,T]$. A stochastic process $X = (X_t)_{t\in[0,T]}$ can also be viewed as a (random) function $X : [0,T] \to \mathbb{R}$, and for given $\omega$ this function is called the path or the trajectory of the process. In this section we introduce the main path properties.

**Definition 2.1.8.** *A function $f : [0,T] \mapsto \mathbb{R}$ is Hölder continuous of order $\alpha$ if there is a constant $C$ such that for every $s,t \in [0,T]$*

$$\sup_{s,t\in[0,T];s\neq t} \frac{|f(t)-f(s)|}{|t-s|^\alpha} \leq C.$$

*The space of Hölder continuous functions on $[0,T]$ is denoted by $C^\alpha([0,T])$.*

The definition of Hölder continuity for stochastic processes is similar.

**Definition 2.1.9.** *A stochastic process $X = (X_t)_{t\in[0,T]}$ is Hölder continuous of order $\alpha$ if there is a finite random variable $C = C(\omega)$ such that for every $s,t \in [0,T]$ we have*

$$\sup_{s,t\in[0,T];s\neq t} \frac{|X_t - X_s|}{|t-s|^\alpha} \leq C(\omega)$$

*almost surely.*

For stochastic processes one can use the following Kolmogorov's continuity theorem (see e.g. [52]) to study the continuity of the process.

**Theorem 2.1.2.** *Let $X = (X_t)_{t\in[0,T]}$ be a stochastic process and assume that there exist positive constants $C$, $\alpha$, and $\beta$ such that*

$$\mathbb{E}|X_t - X_s|^\alpha \leq C|t-s|^{1+\beta}$$

*for every $s,t \in [0,T]$. Then there exists a continuous version of $X$. Moreover, the version is Hölder continuous of any order $a < \frac{\beta}{\alpha}$.*

**Corollary 2.1.1.** *Let $X = (X_t)_{t\in[0,T]}$ be a Gaussian process and assume that there exists a constant $C$ such that*

$$\mathbb{E}[X_t - X_s]^2 \leq C|t-s|^{2\alpha} \tag{2.1}$$

*for every $s, t \in [0, T]$. Then $X$ has a version which is Hölder continuous of any order $a < \alpha$.*

*Proof.* Since $X_t - X_s$ is Gaussian, condition (2.1) implies that for every $p \geq 1$ we have

$$\mathbb{E}[X_t - X_s]^p \leq C_p^p |t-s|^{p\alpha}.$$

Hence by applying Kolmogorov's continuity theorem we obtain that $X$ has a version which is Hölder continuous of any order $a < \alpha - \frac{1}{p}$. Since this holds for every $p \geq 1$ we obtain the result. □

The following *Garsia–Rademich–Rumsey inequality* is also a powerful tool to study the continuity of processes (see [50] and [21]).

**Lemma 2.1.1.** *Let $p \geq 1$ and $\alpha > \frac{1}{p}$. Then there exists a constant $C = C(\alpha, p) > 0$ such that for any continuous function $f$ on $[0, T]$, and for all $0 \leq s, t \leq T$ we have*

$$|f(t) - f(s)|^p \leq C T^{\alpha p - 1} |t-s|^{\alpha p - 1} \int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x-y|^{\alpha p + 1}} \mathrm{d}x \mathrm{d}y.$$

To conclude this section we introduce one more concept of path-regularity.

**Definition 2.1.10.** *The sequence of points $\pi_n = \{0 = t_0^n < t_1^n < \ldots < t_{k(n)}^n = T\}$ on the interval $[0, T]$ is called the partition of the interval $[0, T]$, and the size of the partition is defined as*

$$|\pi_n| = \max_{1 \leq j \leq k(n)} |t_j^n - t_{j-1}^n|.$$

The $p$-variation of a function $f$ along partition $\pi_n$ is defined as

$$v_p(f; \pi_n) = \sum_{t_k \in \pi_n} |\Delta f_{t_k}|^p,$$

where $\Delta f_{t_k} = f_{t_k} - f_{t_{k-1}}$.

**Definition 2.1.11.** *Let $f : [0, T] \mapsto \mathbb{R}$ be a function.*

*1. If the limit*

$$v_p^0(f) = \lim_{|\pi_n| \to 0} v_p(f; \pi_n)$$

*exists, we say that $f$ has finite $p$-variation.*

*2. If*

$$v_p(f) = \sup_{\pi_n} v_p(f; \pi_n) < \infty,$$

*where the supremum is taken over all possible integers $n$ and partitions $\pi_n$, we say that $f$ has bounded $p$-variation.*

We denote by $\mathcal{W}_p([0,T])$ the class of functions with bounded $p$-variation on $[0,T]$ and we equip this class with a norm

$$||f||_{[p]} := (v_p(f))^{\frac{1}{p}} + ||f||_\infty,$$

where $||f||_\infty = \sup_{0\le t\le T}|f(t)|$. It is known that the space $(\mathcal{W}_p, ||\cdot||_{[p]})$ is a Banach space.

## 2.2 Small deviations for Gaussian processes

In this section we briefly introduce some results related to small deviations of Gaussian processes. The main reference in this context is a survey by Li and Shao [37]. See also Lifshits [39] for further reading.

Let $X$ be a Gaussian process on $[0,T]$. The small deviations for Gaussian process $X$ refers to the study of behaviour of small ball probability

$$\mathbb{P}(\sup_{0\le t\le T}|X_t| \le \epsilon)$$

as $\epsilon \to 0$. It is known that in the general setting the behaviour of small ball probability is closely related to the so-called $d$-metric entropy and estimating small ball probability for a Gaussian process $X$ is equivalent to estimating entropy numbers for that process. More precisely, consider centred Gaussian process $X = (X_t)_{t\in T}$ with index set $T$ and the Dudley metric defined by the incremental variance of that process i.e.

$$d(s,t) := [\mathbb{E}|X_s - X_t|^2]^{\frac{1}{2}}, \quad s,t\in T. \tag{2.2}$$

**Definition 2.2.1.** *The entropy number $N(T,d;\epsilon)$ is the minimal number of balls of radius $\epsilon$ that is needed to cover $T$ under Dudley metric defined by (2.2).*

The link between entropy numbers and small ball probabilities is that in general, estimating upper bounds for entropy numbers gives lower bounds for small ball probabilities and vice versa. Hence the lower bounds for small ball probabilities are usually easier to obtain.

Next we give the following general results on the lower bounds. For the proofs, see [37] and references therein.

**Theorem 2.2.1.** *Let $X = (X_t)_{t\in T}$ be a centred Gaussian process. Assume that there is a non-negative function $\phi$ such that $N(T, d; \epsilon) \leq \phi(\epsilon)$, and $c_1\phi(\epsilon) \leq \phi\left(\frac{\epsilon}{2}\right) \leq c_2\phi(\epsilon)$ for some constants $1 < c_1 \leq c_2 < \infty$. Then there exists a constant $C$ such that*

$$\mathbb{P}(\sup_{s,t\in T} |X_s - X_t| \leq \epsilon) \geq exp(-C\phi(\epsilon)).$$

For Gaussian processes with index set $T \subset \mathbb{R}$, the following theorem explains the connection of the incremental variance and the lower bound.

**Theorem 2.2.2.** *Let $X = (X_t)_{t\in[0,1]}$ be a centred Gaussian process with $X_0 = 0$. Assume that there is a function $\sigma(h)$ such that:*

1. *for all $0 \leq s, t \leq 1$ we have $\mathbb{E}(X_s - X_t)^2 \leq \sigma^2(|t - s|)$,*

2. *there are constants $0 < c_1 \leq c_2 < 1$ such that for every $0 \leq h \leq 1$ we have $c_1\sigma(2h \wedge 1) \leq \sigma(h) \leq c_2\sigma(2h \wedge 1)$.*

*Then there is a constant $C$ depending only on $c_1$ and $c_2$ such that*

$$\mathbb{P}(\sup_{0\leq t\leq 1} |X_t| \leq \sigma(\epsilon)) \geq exp\left(-\frac{C}{\epsilon}\right).$$

**Remark 2.2.1.** *The result considers supremum over interval $[0, 1]$. For arbitrary interval one can simply consider a time-changed process.*

As already mentioned, the upper bounds are much harder to obtain and it was pointed out in [37] that the behaviour of the incremental variance is not an appropriate tool to obtain upper bounds. The following concrete example was introduced by Lifshits [40].

**Example 2.2.1.** *Let $\alpha > 0$, $t \in [0, 1]$ and let $\{\xi_k\}_{k\geq 0}$ be a sequence of independent standard normal random variables. Put $\varphi(t) = 1 - |2t - 1|$ and define*

$$X_t = \xi_0 t + \sum_{k=1}^{\infty} 2^{-\frac{\alpha k}{2}} \xi_k \varphi(frac(2^k t)),$$

*where $frac(\cdot)$ denotes the fractional part of a real number. In this case we have a lower bound for incremental variance:*

$$\mathbb{E}(X_s - X_t)^2 \geq c|t - s|^\alpha$$

*for some constant $c > 0$. However, it can be proved that now there are con-*

*stants $C_1$ and $C_2$ such that*

$$exp\left(-C_1\log^2\left(\frac{1}{\epsilon}\right)\right) \le \mathbb{P}(\sup_{0\le t\le 1}|X_t|\le\epsilon) \le exp\left(-C_2\log^2\left(\frac{1}{\epsilon}\right)\right).$$

The example above shows that even if we have a lower bound for incremental variance, we cannot obtain an upper bound for small ball probability similar to Theorem 2.2.2. However, in some cases the upper bound can also be derived. We end the section by giving one result for upper bounds. In Publication III we apply the proof, but the proof is omitted. Hence, for the sake of completeness, we present the key points of the proof here. The proof is essentially taken from Kuelbs et al. [34] and is based on the famous Slepian's lemma (see [57]).

**Lemma 2.2.1.** *Let $X_i$ and $Y_i$ for $i = 1,\dots,n$ be centred Gaussian random variables such that for every $i,j = 1,\dots,n$ we have $\mathbb{E}[X_i^2] = \mathbb{E}[Y_i^2]$ and $\mathbb{E}[X_iX_j] \le \mathbb{E}[Y_iY_j]$. Then for any $x\in\mathbb{R}$ we have*

$$\mathbb{P}(\max_{1\le i\le n} X_i \le x) \le \mathbb{P}(\max_{1\le i\le n} Y_i \le x).$$

**Theorem 2.2.3.** *Assume that the Gaussian process $X$ has stationary increments and define a function by*

$$\sigma(h) := \sqrt{\mathbb{E}[X_{t+h}-X_t]^2}.$$

*If the function $\sigma$ satisfies:*

1. *There exists $\theta\in(0,4)$ such that for every $x\in\left[0,\frac{1}{2}\right]$ we have*

$$\sigma^2(2x) \le \theta\sigma^2(x), \tag{2.3}$$

2. *For every $0<x<1$ and $2\le j\le\frac{1}{x}-2$ we have*

$$6\sigma^2(jx)+\sigma^2((j+2)x)+\sigma^2((j-2)x) \ge 4\sigma^2((j+1)x)+4\sigma^2((j-1)x), \tag{2.4}$$

*then there exists a constant $K>0$ such that for every $x\in(0,1)$ we have*

$$\mathbb{P}\left(\sup_{0\le t\le 1}|X_t-X_0|\le\sigma(x)\right) \le exp\left(-\frac{K}{x}\right).$$

*Proof.* Let $i$ be integer such that $1\le i\le\frac{1}{x}$, $0<x<1$, and define $\xi_i = X_{ix} - X_{(i-1)x}$. Furthermore, for integer $j$ satisfying $1\le j\le\frac{1}{2x}$, set $\eta_j = \xi_{2j}-\xi_{2j-1}$.

It is clear that we have

$$\max_{1\leq i\leq \frac{1}{x}} |\xi_i| \leq 2 \sup_{t\in[0,1]} |X_t - X_0|$$

and

$$\max_{1\leq j\leq \frac{1}{2x}} |\eta_j| \leq 2 \max_{1\leq i\leq \frac{1}{x}} |\xi_i|.$$

As a consequence, we have

$$\begin{aligned} &\mathbb{P}\left(\sup_{0\leq t\leq 1} |X_t - X_0| \leq \sigma(x)\right) \\ &\leq \mathbb{P}\left(\max_{1\leq j\leq \frac{1}{2x}} |\eta_j| \leq 4\sigma(x)\right). \end{aligned}$$

Now by straightforward computations together with assumptions (2.3) and (2.4) we obtain $\mathbb{E}[\eta_j^2] = 4\sigma^2(x) - \sigma^2(2x) \geq (4-\theta)\sigma^2(x)$ and $\mathbb{E}[\eta_i\eta_j] \leq 0$. Denote by $\Phi(x)$ the cumulative distribution function of a standard normal variable. By applying Slepian's Lemma 2.2.1 we get

$$\begin{aligned} &\mathbb{P}\left(\max_{1\leq j\leq \frac{1}{2x}} |\eta_j| \leq 4\sigma(x)\right) \leq \mathbb{P}\left(\max_{1\leq j\leq \frac{1}{2x}} \eta_j \leq 4\sigma(x)\right) \\ &\leq \prod_{j=1}^{\frac{1}{2x}} \mathbb{P}\left(\eta_j \leq 4\sigma(x)\right) = \prod_{j=1}^{\frac{1}{2x}} \Phi\left(\frac{4\sigma(x)}{\sqrt{\mathbb{E}[\eta_j^2]}}\right) \\ &\leq \prod_{j=1}^{\frac{1}{2x}} \Phi\left(\frac{4\sigma(x)}{\sqrt{(4-\theta)\sigma^2(x)}}\right). \end{aligned}$$

It remains to note that

$$\prod_{j=1}^{\frac{1}{2x}} \Phi\left(\frac{4\sigma(x)}{\sqrt{(4-\theta)\sigma^2(x)}}\right) \leq \exp\left(-\frac{K}{x}\right)$$

for some constant $K$. $\square$

## 2.3 Fractional Brownian motion

Standard Brownian motion has stationary and independent increments. Fractional Brownian motion is a generalisation of this by keeping the stationarity of the increments, but allowing them to be dependent which is a useful property in many applications. In this section we present the definition of fractional Brownian motion and list the main properties used in this thesis. For more details on fractional Brownian motion, see book by Biagini et al. [8], Embrechts and Maejima [18], Mishura [43] or Samorodnitsky and Taqqu [55].

**Definition 2.3.1.** *A zero mean Gaussian process $B^H = (B_t^H)_{t\geq 0}$ is a fractional Brownian motion with Hurst index $H \in (0,1)$ if the covariance function $R(s,t)$ is given by*

$$R(s,t) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{2.5}$$

It is not clear from the definition that the covariance function (2.5) is a proper covariance function i.e. it is non-negative definite. For the proof see e.g. [55].

**Remark 2.3.1.** *If $H = \frac{1}{2}$, we have a standard Brownian motion.*

**Theorem 2.3.1.** *Fractional Brownian motion has stationary increments.*

*Proof.* By proposition 3 of section 4 in [38] it is sufficient to show that there is a function $f$ such that $\mathbb{E}|B_t^H - B_s^H|^2 = f(|t-s|)$. This is evident from the definition of fractional Brownian motion. □

**Theorem 2.3.2.** *Fractional Brownian motion is $H$-self-similar.*

*Proof.* Consider centred Gaussian processes $X_t = B_{at}^H$ and $Y_t = a^H B_t^H$. By applying the covariance structure (2.5) of fractional Brownian motion it is straightforward to see that $X$ and $Y$ have the same covariance functions from which the result follows. □

**Remark 2.3.2.** *An equivalent definition for fractional Brownian motion is that it is the unique zero mean $H$-self-similar Gaussian process with stationary increments. Indeed, this implies that the covariance must be of form (2.5).*

**Proposition 2.3.1.** *Let $B^H = (B_t^H)_{t\geq 0}$ be a fractional Brownian motion with $H \neq \frac{1}{2}$ and define the fractional Gaussian noise by*

$$Z_n^H = B_n^H - B_{n-1}^H, \quad n \geq 1.$$

1. *If $H > \frac{1}{2}$, then $Z_n^H$ exhibits long-range dependence,*

2. *if $H < \frac{1}{2}$, then $Z_n^H$ exhibits short-range dependence.*

**Theorem 2.3.3.** *Sample paths of fractional Brownian motion are almost surely Hölder continuous of any order $a < H$.*

*Proof.* This follows directly from Corollary 2.1.1 and the covariance structure (2.5). □

**Remark 2.3.3.** *By Arcones [3] the fractional Brownian motion satisfies the following law of the iterated logarithm:*

$$\mathbb{P}\left(\limsup_{\epsilon\to 0+}\frac{B^H_{t+\epsilon}-B^H_t}{2\epsilon^{2H}\sqrt{\log\log\frac{1}{\epsilon}}}=1\right)=1$$

*for every $t\geq 0$. As a result we obtain that fractional Brownian motion cannot be a-Hölder continuous for any $a\geq H$.*

In what follows we always assume that we have chosen the Hölder continuous version.

**Theorem 2.3.4.** *Fractional Brownian motion is Markovian if and only if $H=\frac{1}{2}$.*

*Proof.* This is a direct consequence of Theorem 2.1.1. □

We end the section by showing that fractional Brownian motion is not a semimartingale for $H\neq\frac{1}{2}$. For this we need the following results of variations for fractional Brownian motion.

**Proposition 2.3.2.** *Let $B^H=(B^H_t)_{t\in[0,T]}$ be a fractional Brownian motion on $[0,T]$. Then*

1. *for every $p>\frac{1}{H}$ we have $v^0_p(B^H)=0$,*

2. *for every $p<\frac{1}{H}$ we have $v_p(B^H)=\infty$.*

*Proof.* 1. Let $\pi$ be any partition of the interval $[0,T]$ and fix $\frac{1}{p}<\alpha<H$. By Hölder continuity of $B^H$ we obtain

$$\sum_{t_k\in\pi}|\Delta B^H_{t_k}|^p\leq C^p\sum_{t_k\in\pi}|\Delta t_k|^{\alpha p}\leq C^p|\pi|^{\alpha p-1}\sum_{t_k\in\pi}|\Delta t_k|.$$

Hence the result follows by letting $|\pi|\to 0$ and noting that $\alpha p>1$.

2. Define equidistant partition by $\tilde{\pi}_n=\{t^n_k=\frac{Tk}{n}:k=1,\dots,n\}$. Then by the self-similarity of fBm we obtain that

$$v_p(B^H;\tilde{\pi}_n)=\sum_{k=1}^n|B^H_{t^n_k}-B^H_{t^n_{k-1}}|^p\overset{\text{law}}{=}T^{pH}n^{1-pH}\frac{1}{n}\sum_{k=1}^n|B^H_k-B^H_{k-1}|^p.$$

Now the result of ergodic theory implies that

$$\frac{1}{n}\sum_{k=1}^{n}|B_k^H - B_{k-1}^H|^p \to \mathbb{E}[B_1^H]^p,$$

where the convergence can be understood almost surely or in $L^2$. Hence for $p < \frac{1}{H}$ we have $v_p(B^H;\tilde{\pi}_n) \to \infty$ which clearly implies $v_p(B^H) = \sup_{\pi_n} v_p(B^H;\pi_n) = \infty$.

□

The following theorem justifies the fact that integration with respect to fractional Brownian motion is not obvious.

**Theorem 2.3.5.** *Fractional Brownian motion is a semimartingale if and only if $H = \frac{1}{2}$.*

*Proof.* Recall that every continuous semimartingale $X$ has finite quadratic variation along suitably chosen sequences. On the other hand, for $H < \frac{1}{2}$ the quadratic variation of $B^H$ does not exist by Proposition 2.3.2. Consequently, it cannot be a semimartingale.

Let now $H > \frac{1}{2}$. We argue by contradiction. Assume that $B^H$ is a semimartingale with decomposition $B^H = M + A$. On the other hand, by Proposition 2.3.2 $B^H$ has zero quadratic variation. As a consequence, the martingale $M$ has zero quadratic variation which in turn implies that it is constant. Thus $B^H = M_0 + A$, and hence $B^H$ has bounded variation. This contradicts Proposition 2.3.2, and hence $B^H$ cannot be a semimartingale for $H > \frac{1}{2}$. □

### 2.3.1 Integral representation of fBm

Fractional Brownian motion can also be represented as an integral of deterministic kernel with respect to standard Brownian motion. There exist several such representations. However, in this thesis we only consider compact interval representation introduced by Molchan and Golosov [45]. See also [29] and [41].

**Definition 2.3.2.** *Define a constant $c_H$ by*

$$c_H = \frac{1}{\Gamma(H+\frac{1}{2})}\left(\frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)}\right)^{\frac{1}{2}},$$

*where $\Gamma$ denotes the Gamma function.*

*1. If $H > \frac{1}{2}$, we set*

$$K_H(t,s) = c_H \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \mathrm{d}u, \quad 0 < s < t < \infty,$$

*and $K_H(t,s) = 0$ otherwise,*

*2. If $H \leq \frac{1}{2}$, we set*

$$K_H(t,s) = c_H \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}}$$
$$- \left(H - \frac{1}{2}\right) c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{3}{2}} \mathrm{d}u, \quad 0 < s < t < \infty,$$

*and $K_H(t,s) = 0$ otherwise.*

Using this kernel we have the following representation.

**Theorem 2.3.6.** *Let $W$ be a standard Brownian motion and $K_H$ as defined above. Then*

$$\int_0^t K_H(t,s) \mathrm{d}W_s \tag{2.6}$$

*defines a fractional Brownian motion with Hurst index $H \in (0,1)$.*

The proof can be found e.g. in [29, 45, 46].

### 2.3.2 Fractional Ornstein-Uhlenbeck processes

In this section we introduce two stochastic processes called fractional Ornstein-Uhlenbeck processes, which are derived from fractional Brownian motion. Both processes are defined as integrals with respect to fractional Brownian motion, and for a moment we take it for granted that such integrals can be defined. The integration with respect to fractional Brownian motion is introduced in Chapter 3. Moreover, for the reader's convenience we refer to [50] on stochastic differential equations driven by fractional Brownian motion. For more details on fractional Ornstein-Uhlenbeck processes, we refer to [30] and [11]. For details on classical Ornstein-Uhlenbeck processes we refer to [18].

Classical Ornstein-Uhlenbeck process can be obtained from standard Brownian motion $W_t$ in the following two ways:

1. $X_t^{(\theta)}$ as a solution of the Langevin equation

$$\mathrm{d}X_t^{(\theta)} = -\theta X_t^{(\theta)} \mathrm{d}t + \mathrm{d}W_t$$

with parameter $\theta > 0$,

2. $\tilde{X}_t^{(\alpha)}$ defined as a Lamperti transform of $W_t$: For $\alpha > 0$ define a process by

$$\tilde{X}_t^{(\alpha)} = e^{-\theta t} W_{\alpha e^{2\theta t}},$$

with parameter $\theta > 0$.

With straightforward computation we can see that for $\alpha = \frac{1}{2\theta}$ (and a suitably chosen initial condition) the processes $X_t^{(\theta)}$ and $\tilde{X}_t^{(\alpha)}$ have the same covariance function and hence they are equivalent in law. Let now $H > \frac{1}{2}$ and replace $W_t$ with fractional Brownian motion $B_t^H$. As a consequence, the processes arising from different definitions are not the same. The process $X_t^{(\theta)}$ as a solution of a Langevin equation $\mathrm{d}X_t^{(\theta)} = -\theta X_t^{(\theta)} \mathrm{d}t + \mathrm{d}B_t^H$ is called the fractional Ornstein-Uhlenbeck process of the first kind. Furthermore, with initial condition $X_0 = \int_{-\infty}^0 e^{\theta s}\, \mathrm{d}\hat{B}_s^H$, where $\hat{B}^H$ is two-sided fractional Brownian motion, the solution can be written as

$$X_t^{(\theta)} = e^{-\theta t} \int_{-\infty}^t e^{\theta s}\, \mathrm{d}\hat{B}_s^H.$$

This process is stationary and exhibits long-range dependence.

On the other hand, define a Lamperti transform of fractional Brownian motion by

$$\tilde{X}_t^{(\alpha)} := e^{-\alpha t} B_{a_t^{(\alpha)}},$$

where $\alpha > 0$ and $a_t^{(\alpha)} = \frac{H}{\alpha} e^{\frac{\alpha t}{H}}$. By defining an auxiliary Gaussian process

$$Y_t^{(\alpha)} = \int_0^t e^{-\alpha s}\, \mathrm{d}B_{a_s^{(\alpha)}}$$

we see that $\tilde{X}^{(\alpha)}$ can be viewed as a solution to $\mathrm{d}\tilde{X}^{(\alpha)} = -\alpha \tilde{X}^{(\alpha)} \mathrm{d}t + \mathrm{d}Y_t^{(\alpha)}$. Furthermore, by noting that $\{Y_{t/\alpha}^{(\alpha)}\}_{t\geq 0} \overset{\text{law}}{=} \{\alpha^{-H} Y_t^{(1)}\}_{t\geq 0}$, we consider Langevin type equation $\mathrm{d}\tilde{X}_t = -\theta \tilde{X}_t \mathrm{d}t + \mathrm{d}Y_t^{(1)}$ with some parameter $\theta > 0$. The solution to this is referred to as the fractional Ornstein-Uhlenbeck process of the second kind. Furthermore, special selection $X_0 = \int_{-\infty}^0 e^{(\theta-1)s}\, \mathrm{d}B_{a_s^{(1)}}$ leads to a unique solution

$$\tilde{X}_t = e^{-\theta t} \int_{-\infty}^t e^{(\theta-1)s}\, \mathrm{d}B_{a_s^{(1)}}.$$

This process is stationary and exhibits short-range dependence.

# 3. Integration with respect to Gaussian processes

Many stochastic processes of interest are not semimartingales and hence the classical Itô integration theory cannot be applied. In particular, fractional Brownian motion is not a semimartingale. In this section we introduce two approaches to define integrals with respect to Gaussian processes. The first approach is pathwise integrals, which are under study on Publication I, Publication II, and Publication III. Another approach to define integrals with respect to Gaussian processes is the so-called Skorokhod integral or divergence integral, which is defined as an adjoint operator of the Malliavin derivative. The tools provided by Malliavin calculus are applied in Publication IV.

## 3.1 Pathwise integrals

In this section we introduce three kinds of pathwise integrals: Young integrals, Föllmer integrals, and generalised Lebesgue–Stieltjes integrals. For further reading, see also [12, 13, 16, 20] on pathwise derivation and pathwise functional calculus.

### 3.1.1 Young integral

A contributing work by Young [63] extended classical Riemann-Stieltjes to cover functions of unbounded variation. More precisely, he noticed that $p$-variations can be useful to define integrals. For proofs see also [53].

**Theorem 3.1.1.** *Let $f \in \mathcal{W}_p([0,T])$ and $g \in \mathcal{W}_q([0,T])$ for some $1 \leq p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} > 1$. Moreover, assume that $f$ and $g$ have no common points of discontinuities. Then for any interval $[s,t] \subset [0,T]$ the integral*

$$\int_s^t f \mathrm{d}g$$

*exists as a Riemann-Stieltjes integral.*

Zähle [64] considered Hölder continuous functions and showed that also in this case the Riemann-Stieltjes integral exists if $f$ and $g$ are together "smooth enough". More precisely, he proved the following:

**Theorem 3.1.2.** *Let $f \in C^{\alpha}([0,T])$ and $g \in C^{\beta}([0,T])$. If $\alpha+\beta>1$, then for any interval $[s,t] \subset [0,T]$ the integral*

$$\int_s^t f \mathrm{d}g$$

*exists as a Riemann-Stieltjes integral.*

### 3.1.2 Föllmer integral

In applications, especially in financial mathematics, it is a wanted feature to define stochastic integrals as a limit of Riemann-Stieltjes sums, or so-called forward integrals. In this section we introduce Föllmer integrals, and for the results of this section we refer especially to the original paper by Hans Föllmer [19]. See also Sondermann [58].

**Definition 3.1.1.** *Let $(\pi_n)_{n=1}^{\infty}$ be a sequence of partitions $\pi_n = \{0 = t_0^n < \ldots < t_{k(n)}^n = T\}$ such that $|\pi_n| = \max_{j=1,\ldots,k(n)} |t_j^n - t_{j-1}^n| \to 0$ as $n \to \infty$ and let $X = (X_t)_{t\in[0,T]}$ be a continuous process. The Föllmer integral of a process $Y$ with respect to $X$ over interval $[0,t]$ along the sequence $\pi_n$ is defined as*

$$\int_0^t Y_s \mathrm{d}X_s = \lim_{n\to\infty} \sum_{t_j^n \in \pi_n \cap (0,t]} Y_{t_{j-1}}^n \left( X_{t_j^n} - X_{t_{j-1}^n} \right)$$

*if the limit exists almost surely. The integral over the whole interval $[0,T]$ is defined as*

$$\int_0^T Y_s \mathrm{d}X_s = \lim_{t\to T} \int_0^t Y_s \mathrm{d}X_s.$$

**Remark 3.1.1.** *If the processes $X$ and $Y$ are Hölder continuous processes of order $\alpha$ and $\beta$ with $\alpha+\beta>1$, then it can be shown that the Föllmer integral exists and coincides with the Young integral.*

We also remark that while the definition is very useful for applications, it can sometimes be difficult to show that the Föllmer integral exists. However, in some cases the existence of the Föllmer integral can be proved. For instance, this is the case for processes $X$ that have finite quadratic variation. We first recall the definition of a quadratic variation process.

**Definition 3.1.2.** *Let $(\pi_n)_{n=1}^{\infty}$ be a sequence of partitions $\pi_n = \{0 = t_0^n < \ldots < t_{k(n)}^n = T\}$ such that $|\pi_n| = \max_{j=1,\ldots,k(n)} |t_j^n - t_{j-1}^n| \to 0$ as $n \to \infty$. Let*

*$X$ be a continuous process. Then $X$ is a* quadratic variation process along the sequence $(\pi_n)_{n=1}^\infty$ *if the limit*

$$\langle X\rangle_t = \lim_{n\to\infty} \sum_{t_j^n\in\pi_n\cap(0,t]} \left(X_{t_j^n} - X_{t_{j-1}^n}\right)^2$$

*exists almost surely.*

**Lemma 3.1.1.** *[19] Let $X$ be a continuous quadratic variation process and let $f \in C^{1,2}([0,T]\times\mathbb{R})$. Let $0 \le s < t \le T$. Then*

$$\begin{aligned} f(t,X_t) &= f(s,X_s) + \int_s^t \frac{\partial f}{\partial t}(u,X_u)\,\mathrm{d}u + \int_s^t \frac{\partial f}{\partial x}(u,X_u)\,\mathrm{d}X_u \\ &\quad + \frac{1}{2}\int_s^t \frac{\partial^2 f}{\partial x^2}(u,X_u)\,\mathrm{d}\langle X\rangle_u. \end{aligned}$$

*In particular, the Föllmer integral $\int_s^t \frac{\partial f}{\partial x}(u,X_u)\,\mathrm{d}X_u$ exists and has a continuous modification.*

For the proof and details, see [19] or [58].

### 3.1.3 Generalised Lebesgue-Stieltjes integral

In this subsection we briefly introduce the concept of the generalized Lebesgue-Stieltjes integral which is a powerful tool for studying existence of integrals.

The generalized Lebesgue–Stieltjes integral is based on fractional integration and fractional Besov spaces. For details on fractional integration we refer to [54] and for fractional Besov spaces we refer to [50]. See also [43] and [51].

**Definition 3.1.3.** *Let $f \in L^1([a,b])$ and $t \in (a,b)$. The* Riemann–Liouville fractional integrals *$I_{a+}^\beta$ and $I_{b-}^\beta$ of order $\beta\in(0,1)$ are defined as*

$$\begin{aligned} (I_{a+}^\beta f)(t) &= \frac{1}{\Gamma(\beta)}\int_a^t f(u)(t-u)^{\beta-1}\,\mathrm{d}u, \\ (I_{b-}^\beta f)(t) &= \frac{(-1)^{-\beta}}{\Gamma(\beta)}\int_t^b f(u)(u-t)^{\beta-1}\,\mathrm{d}u, \end{aligned}$$

*where $\Gamma$ is the Gamma-function. The* Riemann–Liouville fractional derivatives *$D_{a+}^\beta$ and $D_{b-}^\beta$ of order $\beta\in(0,1)$ are defined as*

$$\begin{aligned} (D_{a+}^\beta f)(t) &= \frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f(u)(t-u)^{-\beta}\mathrm{d}u, \\ (D_{b-}^\beta f)(t) &= \frac{-1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^b f(u)(u-t)^{-\beta}\mathrm{d}u. \end{aligned}$$

Assume that two functions $f, g : [a,b] \mapsto \mathbb{R}$ are such that the right limit

$f(t+)$ and the left limit $g(s-)$ exist for every $t \in [a,b)$ and $s \in (a,b]$, and set $f_{a+}(t) = (f(t) - f(a+))\mathbf{1}_{(a,b)}(t)$, and $g_{b-}(t) = (g(b-) - g(t))\mathbf{1}_{(a,b)}(t)$.

**Definition 3.1.4.** *Let $f$ and $g$ be such that $f_{a+} \in I^{\alpha}_{a+}(L^p([a,b]))$ and $g_{b-} \in I^{1-\alpha}_{b-}(L^q([a,b]))$ for some $\alpha \in (0,1)$ and $p,q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$. Then the generalised Lebesgue-Stieltjes integral over interval $[a,b]$ is defined as*

$$\int_a^b f \,\mathrm{d}g = \int_a^b (D^{\beta}_{a+} f_{a+})(s)(D^{1-\beta}_{b-} g_{b-})(s)\,\mathrm{d}s + f(a+)(g(b-) - g(a+)).$$

*Furthermore, the integral is independent of $\beta$.*

**Remark 3.1.2.** *We list some properties of generalised Lebesgue-Stieltjes integrals which are not evident from the definition. In the following formulas we assume that all the integrals exist.*

- $\int_a^c f \mathrm{d}g + \int_c^b f \mathrm{d}g = \int_a^b f \mathrm{d}g$,
- *for $(c,d) \subset [a,b]$ we have $\int_a^b \mathbf{1}_{(c,d)} f \mathrm{d}g = \int_c^d f \mathrm{d}g$,*
- $\int_a^b \mathbf{1}_{(c,d]} \mathrm{d}g = g(d) - g(c)$, *provided that $g$ is Hölder continuous,*
- *if $f_1 = f_2$ almost everywhere, then $\int_a^b f_1 \mathrm{d}g = \int_a^b f_2 \mathrm{d}g$.*

Next we recall the definition of fractional Besov spaces on $[0,T]$.

**Definition 3.1.5.** *Fix $0 < \beta < 1$.*

1. *The* fractional Besov space $W^{\beta}_1 = W^{\beta}_1([0,T])$ *is the space of real-valued measurable functions $f : [0,T] \to \mathbb{R}$ such that*

$$\|f\|_{1,\beta} = \sup_{0 \leq s < t \leq T} \left( \frac{|f(t) - f(s)|}{(t-s)^{\beta}} + \int_s^t \frac{|f(u) - f(s)|}{(u-s)^{1+\beta}} \,\mathrm{d}u \right) < \infty.$$

2. *The* fractional Besov space $W^{\beta}_2 = W^{\beta}_2([0,T])$ *is the space of real-valued measurable functions $f : [0,T] \to \mathbb{R}$ such that*

$$\|f\|_{2,\beta} = \int_0^T \frac{|f(s)|}{s^{\beta}} \,\mathrm{d}s + \int_0^T \int_0^s \frac{|f(s) - f(u)|}{(s-u)^{1+\beta}} \,\mathrm{d}u\mathrm{d}s < \infty.$$

Note that $\|f\|_{1,\beta}$ is only a seminorm.

**Remark 3.1.3.** *Consider the space $C^\alpha = C^\alpha([0,T])$ of Hölder continuous functions of order $\alpha$ and let $0 < \epsilon < \beta \wedge (1-\beta)$. Then we have the following inclusions:*

$$C^{\beta+\epsilon} \subset W_1^\beta \subset C^{\beta-\epsilon} \quad \textit{and} \quad C^{\beta+\epsilon} \subset W_2^\beta.$$

For $f \in W_1^\beta[0,T]$ the restriction of $f$ to $[0,t] \subset [0,T]$ belongs to $I_{0+}^\beta(L^1[0,t])$. Similarly, for $g \in W_2^\beta[0,T]$ the restriction of $g$ to $[0,t]$ belongs to $I_{t-}^{1-\beta}(L^\infty[0,t])$. Hence we have the following proposition.

**Proposition 3.1.1.** *[50] Let $0 < \beta < 1$ and let $f \in W_2^\beta([0,T])$ and $g \in W_1^{1-\beta}([0,T])$. Then for any interval $[a,b] \subset [0,T]$ the* generalized Lebesgue–Stieltjes integral

$$\int_a^b f\,\mathrm{d}g = \int_a^b (D_{a+}^\beta f_{a+})(s)(D_{b-}^{1-\beta} g_{b-})(s)\,\mathrm{d}s + f(a+)(g(b-) - g(a+))$$

*exists.*

**Remark 3.1.4.** *It is shown in [64] that if the processes $X$ and $Y$ are Hölder continuous processes of order $\alpha$ and $\beta$ with $\alpha + \beta > 1$, then the generalized Lebesgue-Stieltjes integral exists and it coincides with the Young integral. Hence, according to Remark 3.1.1, in this case the integral can be understood as the Young integral, the Föllmer integral or the generalised Lebesgue-Stieltjes integral. Moreover, all integrals coincide.*

We will also use the following estimate many times.

**Theorem 3.1.3.** *[50] Let $f \in W_2^\beta([0,T])$ and $g \in W_1^{1-\beta}([0,T])$. Then for any $t \in (0,T]$ we have the estimate*

$$\left| \int_0^t f\,\mathrm{d}g \right| \leq \sup_{0 \leq s < t \leq T} \left| D_{t-}^{1-\beta} g_{t-}(s) \right| \|f\|_{2,\beta}.$$

**Corollary 3.1.1.** *Let $f, f_n \in W_2^\beta([0,T])$ and $g \in W_1^{1-\beta}([0,T])$. If $||f - f_n||_{2,\beta} \to 0$, then*

$$\int_0^t f_n \mathrm{d}g \to \int_0^t f \mathrm{d}g.$$

*for every $t \in (0,T]$.*

## 3.2 Divergence integral and Malliavin calculus

In this section we briefly introduce Malliavin calculus with respect to certain Gaussian processes; in particular, for fractional Brownian motion. For more details, we refer to [2], [48] and [47].

Let $W$ be a standard Brownian motion and assume $G = (G_t)_{t\in[0,T]}$ is a continuous centred Gaussian process of the form

$$G_t = \int_0^t K(t,s)\mathrm{d}W_s \tag{3.1}$$

where the kernel $K$ satisfies $\sup_{t\in[0,T]}\int_0^t K(t,s)^2\mathrm{d}s < \infty$. In particular, the fractional Brownian motion is of this form by representation (2.6). First we recall some definitions.

**Definition 3.2.1.** *We denote by $\mathcal{E}_G$ the set of simple random variables of the form*

$$F = \sum_{k=1}^n a_k G_{t_k}$$

*where $n \in \mathbb{N}, a_k \in \mathbb{R}$ and $t_k \in [0,T]$ for $k = 1,\dots,n$.*

**Definition 3.2.2.** *The Gaussian space $\mathcal{H}_1$ associated to $G$ is the closure of $\mathcal{E}_G$ in $L^2(\Omega)$.*

**Definition 3.2.3.** *The reproducing Hilbert space $\mathcal{H}_G$ of $G$ is the closure of $\mathcal{E}_G$ with respect to the inner product*

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\rangle_{\mathcal{H}} = R_G(t,s).$$

In what follows we will drop $G$ on the notation.

It is known that the mapping $\mathbf{1}_{[0,t]} \mapsto G_t$ can be extended to an isometry between the Hilbert space $\mathcal{H}$ and the Gaussian space $\mathcal{H}_1$. The image of $\varphi \in \mathcal{H}$ in this isometry is denoted by $G(\varphi)$. In particular, we have $G(\mathbf{1}_{[0,t]}) = G_t$.

**Definition 3.2.4.** *Denote by $\mathcal{S}$ the space of all smooth random variables of the form*

$$F = f(G(\varphi_1),\cdots,G(\varphi_n)), \qquad \varphi_1,\cdots,\varphi_n \in \mathcal{H},$$

*where $f \in C_b^\infty(\mathbb{R}^n)$ i.e. $f$ and all its derivatives are bounded. The Malliavin derivative $D = D^{(G)}$ of $F$ is an element of $L^2(\Omega;\mathcal{H})$ defined by*

$$DF = \sum_{i=1}^n \partial_i f(G(\varphi_1),\cdots,G(\varphi_n))\varphi_i.$$

*In particular, $DG_t = \mathbf{1}_{[0,t]}$.*

**Definition 3.2.5.** *Let $\mathbb{D}_G^{1,2} = \mathbb{D}^{1,2}$ be the Hilbert space of all square integrable Malliavin derivative random variables defined as the closure of $\mathcal{S}$ with respect* *to norm*

$$\|F\|_{1,2}^2 = \mathbb{E}|F|^2 + \mathbb{E}(\|DF\|_{\mathcal{H}}^2).$$

Now we are ready to define the divergence operator $\delta$ as the adjoint operator of the Malliavin derivative $D$.

**Definition 3.2.6.** *The domain Dom $\delta$ of the operator $\delta$ is the set of random variables $u \in L^2(\Omega; \mathcal{H})$ satisfying*

$$\left|\mathbb{E}\langle DF, u\rangle_{\mathcal{H}}\right| \leq c_u \|F\|_{L^2}$$

*for any $F \in \mathbb{D}^{1,2}$ and some constant $c_u$ depending only on $u$. For $u \in$ Dom $\delta$ the divergence operator $\delta(u)$ is a square integrable random variable defined by the duality relation*

$$\mathbb{E}(F\delta(u)) = \mathbb{E}\langle DF, u\rangle_{\mathcal{H}}, \qquad \forall F \in \mathbb{D}^{1,2}$$

*for any $F \in \mathbb{D}^{1,2}$.*

**Remark 3.2.1.** *It is well-known that $\mathbb{D}^{1,2} \subset$ Dom $\delta$.*

We use the notation

$$\delta(u) = \int_0^T u_s \delta G_s.$$

**Example 3.2.1.** *In the case of standard Brownian motion, we have $\mathcal{H}_W = L^2([0,T])$, and in this case we have*

$$\delta(\varphi) = W(\varphi) = \int_0^T \varphi(s) \mathrm{d}W_s.$$

**Remark 3.2.2.** *In general, the divergence is an extension of the Itô integral. Moreover, it coincides with the stochastic integral introduced by Skorokhod [56]. For this reason the divergence operator is also referred to as the Skorokhod integral or the divergence integral.*

Recall now the special form of $G$ given by (3.1) and define a linear operator $K^*$ from $\mathcal{E}$ to $L^2[0,T]$ by

$$(K^*\varphi)(s) = \varphi(s)K(T,s) + \int_s^T \left[\varphi(t) - \varphi(s)\right] K(\mathrm{d}t, s).$$

With the help of this operator, the Hilbert space $\mathcal{H}$ generated by $G$ can be represented as $\mathcal{H} = (K^*)^{-1}(L^2[0,T])$. Furthermore, $\mathbb{D}_G^{1,2}(\mathcal{H}) = (K^*)^{-1}\big(\mathbb{D}_W^{1,2}(L^2[0,T])\big)$. Moreover, we can represent $\delta^{(G)}$ with $\delta^{(W)}$ by the relation

$$\int_0^t u_s \delta G_s = \int_0^t (Ku)_s \delta W_s$$

provided that $Ku \in \operatorname{Dom} \delta^{(W)}$. Similarly, by considering iterated integrals with respect to Gaussian process $G$ one can define $n$th Wiener chaos as the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n\left(G(\varphi)\right), \varphi \in \mathcal{H}, \|\varphi\|_{\mathcal{H}} = 1\}$, where $H_n$ is the $n$th Hermite polynomial. In this case we have that the mapping $I_n^G(\varphi^{\otimes n}) = n! H_n\left(G(\varphi)\right)$ provides a linear isometry between the symmetric tensor product $\mathcal{H}^{\otimes n}$ space and subspace $\mathscr{H}_n$. For details on Malliavin calculus for multiple stochastic integrals with general Gaussian processes we refer to [49].

We end the section with the following proposition (taken from [49]) which provides a central limit theorem for a sequence of multiple Wiener integrals. We apply this result in Publication IV.

**Proposition 3.2.1.** *Let $\{F_n\}_{n\geq 1}$ be a sequence of random variables in the $q$th Wiener chaos, $q \geq 2$, such that $\lim_{n\to\infty} \mathbb{E}(F_n^2) = \sigma^2$. Then the following statements are equivalent:*

**(i)** *$F_n$ converges in distribution to $\mathcal{N}(0, \sigma^2)$ as $n$ tends to infinity.*

**(ii)** *$\|DF_n\|^2_{\mathcal{H}}$ converges in $L^2(\Omega)$ to $q\sigma^2$ as $n$ tends to infinity.*

# 4. Summaries of the articles

## I. Rate of convergence for discretization of integrals with respect to Fractional Brownian motion

Let $B^H$ be a fractional Brownian motion with $H > \frac{1}{2}$ and let $f$ be a convex function. It was shown in [4] that the integral

$$S := \int_0^T f'_-(B^H_u)\mathrm{d}B^H_u$$

exists almost surely in the sense of the generalised Lebesgue-Stieltjes integral. Moreover, the integral can be understood as a limit of Riemann–Stieltjes sums i.e.

$$S_n := \sum_{i=1}^n f'_-(B^H_{\frac{i-1}{n}})(B^H_{\frac{i}{n}} - B^H_{\frac{i-1}{n}}) \to S \tag{4.1}$$

where the limit is understood almost surely. In this article we show that under an additional assumption on the convex function $f$, the convergence (4.1) holds also in $L^r$ for a suitable range of $r$. Moreover, we determine the rate of convergence. More precisely, let

$$2H < p < \frac{H}{1-H},$$

$$1 - H < \beta < \frac{H}{p},$$

and define a function $C : \mathbb{R} \to \mathbb{R}$ by

$$C(a) = \max(1, |a|)e^{-\frac{\min\{a^2,(a-1)^2\}}{2}}.$$

Let $\mu$ denote the measure associated to the second derivative $f''$. We prove that if

$$\int_{\mathbb{R}} C(a)^{\frac{1}{p}}\mu(\mathrm{d}a) < \infty,$$

then for every $r \in [1, p)$ there exists a constant $C = C(f, r, p, H, \beta)$ such that

$$(\mathbb{E}|S_n - S|^r)^{\frac{1}{r}} \leq C \left(\frac{1}{n}\right)^{\frac{H}{p} - \beta}.$$

As a consequence, the rate of convergence can be brought arbitrary close to $H - \frac{1}{2}$ by letting $\beta \to 1 - H$ and $p \to 2H$. For the result we derive an upper bound for the crossing probability $\mathbb{P}(B_s^H < a < B_t^H)$ for fractional Brownian motion, which is an improvement of a similar upper bound derived in [14].

## II Pathwise Integrals and Itô–Tanaka Formula for Gaussian Processes

Let $B^H$ be a fractional Brownian motion with $H > \frac{1}{2}$ and let $f$ be a convex function. In [4] it was proved that the following Itô formula

$$F(B_T^H) = F(0) + \int_0^T f'_-(B_u^H) \mathrm{d}B_u^H$$

holds where the integral is understood as a generalised Lebesgue–Stieltjes integral. Moreover, one can understand the integral as a limit of Riemann–Stieltjes sums. In this article we extend this result for wider class of Gaussian processes. More precisely, we consider the following class of Gaussian processes:

**Definition 1.** *A centred continuous Gaussian process* $X = (X_t)_{t \in [0,T]}$ *with the covariance* $R$ *belongs to the* class $\mathcal{X}^\alpha$ *if*

1. $R(s,t) > 0$ *for every* $s, t > 0$,

2. *the "worst case" incremental variance satisfies, at* $t = 0$,

$$\sup_{0 \leq s \leq T-t} W(t+s, s) = Ct^{2\alpha} + o(t^{2\alpha}),$$

   *where* $C > 0$ *and* $0 < \alpha < 1$,

3. *there exist* $c, \delta > 0$ *such that*

$$V(s) \geq cs^2,$$

   *when* $s \leq \delta$,

*4. there exists a* $\delta > 0$ *such that*

$$\sup_{0<t<2\delta} \sup_{\frac{t}{2}\leq s\leq t} \frac{R(s,s)}{R(t,s)} < \infty.$$

The definition is rather technical. However, the assumptions are not very restrictive and are satisfied for many Gaussian processes (For examples and remarks, see Publication II).

We also derive an Itô–Tanaka formula for Gaussian processes of form $Y = X + M$, where $M$ is a Gaussian martingale and $X \in \mathcal{X}^\alpha$. In particular, we show the existence of the Föllmer integral in the Itô–Tanaka formula. To obtain such a result we generalise the upper bound for crossing probability $\mathbb{P}(X_s < a < X_t)$ derived in Publication I for fractional Brownian motion.

## III. Integral representation of random variables with respect to Gaussian processes

Consider probability space $(\Omega, \mathbf{F}, \mathbb{P})$ and let $W$ be a standard Brownian motion in that space. Dudley [15] showed that for any $\mathcal{F}_1$-measurable random variable $\xi$ there exists an adapted process $\phi$ such that

$$\xi = \int_0^1 \phi(s)\mathrm{d}W_s.$$

Later on, Mishura et al. [44] showed that this is also true if standard Brownian motion is replaced with fractional Brownian motion $B^H$ and the integral is understood in the pathwise sense. Motivated by these we generalise the result to wider class of Gaussian processes. In particular, we consider class of Gaussian processes similar to $\mathcal{X}^\alpha$ defined in Publication II and show that only local properties of the covariance structure play a role in obtaining such results. More precisely, we prove that if the assumptions of $X \in \mathcal{X}^\alpha$ holds for incremental process $Y_t = X_{t+u} - X_u$ for $u, t$ close to some fixed $T$, we have:

1. For any distribution function $F$ there exists an adapted process $\phi_T$ such that $\int_0^T \phi_T(s)\mathrm{d}X_s$ has distribution $F$,

2. For any $\mathcal{F}_T$-adapted random variable $\xi$ there exists an adapted process $\Psi_T$ such that

$$\xi = \lim_{t\to T-} \int_0^t \Psi_T(s)\mathrm{d}X_s,$$

3. if in addition $\xi$ is an end value of a Hölder continuous process of order $a > 1 - \alpha$, then $\xi$ can be represented as a proper integral i.e.

$$\xi = \int_0^T \Psi_T(s) \mathrm{d}X_s,$$

To prove our results we apply the Itô formula and an estimate for the crossing probability derived in Publication II. Moreover, we apply small ball probabilities for Gaussian processes $X \in \mathcal{X}^\alpha$. For small ball probabilities we simply assume that a given Gaussian process satisfies a certain upper bound. For justification we show that a wide class of processes indeed satisfy the given assumption.

## IV. Parameters estimation based on discrete observations of fractional Ornstein-Uhlenbeck process of the second kind

In mathematical statistics it is of great interest to estimate a certain unknown parameter of the system from observations. In other words, the aim is to approximate the unknown. In this article we assume that we have observed a fractional Ornstein-Uhlenbeck process of the second kind described in subsection 2.3.2 at discrete time points and we provide strong consistent estimators for unknown parameters $H$ and $\theta$. Moreover, we prove that our estimators are asymptotically Gaussian and we find the rate of convergence.

Let $X$ be a fractional Ornstein-Uhlenbeck process of the second kind and put

$$\Psi(\theta) = \frac{(2H-1)H^{2H}}{\theta} B((\theta - 1)H + 1, 2H - 1) \tag{4.2}$$

where $B(x, y)$ denotes the Beta function. As our main theorem we prove the following:

**Theorem 1.** *Assume we observe $X_t$ at discrete points $\{t_k = k\Delta_N, k = 0, 1, \ldots, N\}$ and $T_N = N\Delta_N$. Assume we have $\Delta_N \to 0$, $T_N \to \infty$ and $N\Delta_N^2 \to 0$ as $N$ tends to infinity. Put*

$$\widehat{\mu}_{2,N} = \frac{1}{T_N} \sum_{k=1}^{N} X_{t_k}^2 \Delta t_k \quad \textit{and} \quad \widehat{\theta}_N := \Psi^{-1} \left( \widehat{\mu}_{2,N} \right) \tag{4.3}$$

*where $\Psi^{-1}$ is the inverse of function $\Psi$ given in (4.2). Then $\widehat{\theta}$ is a strongly consistent estimator of the drift parameter $\theta$ in the sense that as $N$ tends to infinity, we have*

$$\widehat{\theta}_N \longrightarrow \theta \tag{4.4}$$

*almost surely. Moreover, as $N$ tends to infinity, we have*

$$\sqrt{T_N}(\widehat{\theta}_N - \theta) \xrightarrow{d} \mathcal{N}(0, \sigma_\theta^2) \tag{4.5}$$

*where*

$$\sigma_\theta^2 = \frac{\sigma^2}{[\Psi'(\theta)]^2}. \tag{4.6}$$

We also give a formula for $\sigma^2$. The asymptotic Gaussianity for our estimator $\widehat{\theta}_N$ is based on the machinery of Malliavin calculus described in section 3.2.

For the Hurst parameter $H$ we give an estimator which is based on a method of generalised quadratic variations introduced by Istas and Lang [27]. Furthermore, we introduce an estimator $\tilde{\theta}_N$ similar to $\widehat{\theta}_N$ given in (4.3) in a case where also $H$ is unknown.

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# Errata

## Publication I

1. On p. 24 the right side of the first equation "$= -\frac{1}{2\sigma^2}\big[\big(y - a\frac{R(t,s)}{R(s,s)}\bar{\sigma}^2..$" should be "$= -\frac{1}{2\bar{\sigma}^2}\big[\big(y - a\frac{R(t,s)}{R(s,s)}\bar{\sigma}^2..$"

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