Characterizations and fine properties of functions of bounded variation on metric measure spaces

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Abstract

   In this thesis we study functions of bounded variation, abbreviated as BV functions, on metric measure spaces. We always assume the space to be equipped with a doubling measure, and mostly we also assume it to support a Poincaré inequality.

A central topic in the thesis are the various characterizations of BV functions. We show that BV functions can be characterized by a pointwise inequality involving the maximal function of a finite measure. Furthermore, we study the Federer-type characterization of sets of finite perimeter, according to which a set is of finite perimeter if and only if the codimension one Hausdorff measure of the set's measure theoretic boundary is finite. Through the study of so-called strong relative isoperimetric inequalities, we establish a slightly weakened version of this characterization.

Moreover, we prove the Federer-type characterization on spaces that support a geometric Semmes family of curves. On such spaces, between every pair of points there is a curve family with certain uniformity properties that resemble the behavior of parallel lines on a Euclidean space. Our proof relies on first proving a characterization of BV functions in terms of curves.

We also study functionals of linear growth, which give a generalization of BV functions. We consider an integral representation for such functionals by means of the variation measure, but contrary to the Euclidean case, the functional and the integral representation are only comparable instead of being equal. As a by-product of our analysis, we are able to characterize those BV functions that are in fact Newton-Sobolev functions.

   As an application of the integral representation, we consider a minimization problem for the functionals of linear growth, and show that the boundary values of such a problem can be expressed as a penalty term in which we integrate over the boundary of the domain. For this, we need to study boundary traces and extensions of BV functions. Our analysis of traces also produces novel pointwise results on the behavior of BV functions in their jump sets.

Tiivistelmä

Tässä väitöskirjassa tutkitaan rajoitetusti heilahtelevia funktioita, lyhennettynä BV-funktioita, metrisissä mitta-avaruuksissa. Avaruuden oletetaan aina olevan varustettu tuplaavalla mitalla, ja enimmäkseen oletetaan myös, että Poincarén epäyhtälö pätee.

Keskeinen aihe väitöskirjassa ovat BV-funktioiden erilaiset karakterisaatiot. Työssä osoitetaan, että BV-funktiot voidaan karakterisoida pisteittäisellä epäyhtälöllä, jossa esiintyy äärellisen mitan maksimaalifunktio. Lisäksi tutkitaan äärellisperimetristen joukkojen Federer-tyyppistä karakterisaatiota, jonka mukaan joukolla on äärellinen perimetri jos ja vain jos joukon mittateoreettisen reunan kodimensio yksi Hausdorffin mitta on äärellinen. Työssä todistetaan lievästi heikennetty versio tästä karakterisaatiosta tutkimalla niin sanottuja vahvoja relatiivisia isoperimetrisiä epäyhtälöitä.

Lisäksi osoitetaan, että yllä mainittu Federer-tyyppinen karakterisaatio pätee sellaisissa avaruuksissa, joissa esiintyy geometrinen Semmesin polkuparvi. Tällaisissa avaruuksissa jokaisen kahden pisteen välillä on polkuparvi, jolla on tiettyjä uniformisuusominaisuuksia, jotka jäljittelevät yhdensuuntaisten suorien käyttäytymistä euklidisessa avaruudessa. Federer-tyyppisen karakterisaation todistus nojautuu käyrien avulla esitettyyn BV-funktioiden karakterisaatioon, joka todistetaan ensin.

Työssä tutkitaan myös lineaarisen kasvun funktionaaleja, jotka muodostavat BV-funktioiden yleistyksen. Tällaisille funktionaaleille tarkastellaan integraaliesitystä variaatiomitan avulla, mutta toisin kuin euklidisessa tapauksessa, funktionaali ja integraaliesitys ovat vain verrannolliset eivätkä yhtäsuuret. Tarkastelujen sivutuotteena onnistutaan karakterisoimaan sellaiset BV-funktiot, jotka ovat Newton-Sobolev-funktioita.

Integraaliesityksen sovelluksena tutkitaan lineaarisen kasvun funktionaaleihin liittyvää minimointiongelmaa ja osoitetaan, että tällaisen ongelman reuna-arvot voidaan esittää sakkoterminä, jossa integroidaan alueen reunan yli. Tätä varten on tutkittava BV-funktioiden reunajälkiä ja jatkeita. Työssä esitetty reunajälkien analyysi johtaa myös uusiin pisteittäisiin tuloksiin koskien BV-funktioiden käyttäytymistä niiden hyppyjoukossa.

Preface

First of all, I wish to express my gratitude to my supervisor, professor Juha Kinnunen for his guidance and advice throughout my doctoral studies. Since the time I first embarked on the study of mathematical analysis in 2009, he has provided me with abundant personal instruction, worthwhile research topics, as well as opportunities for domestic and international collaboration with various experts in the field.

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Helsinki, April 4, 2014,

Panu Lahti

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List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I Panu Lahti and Heli Tuominen. A pointwise characterization of functions of bounded variation on metric spaces.
Accepted for publication in Ricerche di Matematica,
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II Riikka Korte and Panu Lahti. Relative isoperimetric inequalities and sufficient conditions for finite perimeter on metric spaces.
Annales de l'Institut Henri Poincaré Analyse Non Linéaire, Volume 31 (2014), Issue 1, Pages 129–154,
DOI 10.1016/j.anihpc.2013.01.005,
February 2014.

III Riikka Korte, Panu Lahti, and Nageswari Shanmugalingam. Semmes family of curves and a characterization of functions of bounded variation in terms of curves.
http://cvgmt.sns.it/paper/2229/,
September 2013.

IV Heikki Hakkarainen, Juha Kinnunen, Panu Lahti, and Pekka Lehtelä. Relaxation and integral representation for functionals of linear growth on metric measure spaces.
arXiv:1401.5717,
January 2014.

V Panu Lahti. Extensions and traces of functions of bounded variation on metric spaces.
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Author's Contribution

Publication I: "A pointwise characterization of functions of bounded variation on metric spaces"

The author has performed a substantial part of the analysis and writing of the article.

Publication II: "Relative isoperimetric inequalities and sufficient conditions for finite perimeter on metric spaces"

The author has played a central role in preparing the article.

Publication III: "Semmes family of curves and a characterization of functions of bounded variation in terms of curves"

The author has performed a substantial part of the analysis and writing of the article.

Publication IV: "Relaxation and integral representation for functionals of linear growth on metric measure spaces"

The author has played a central role in preparing the article.

Publication V: "Extensions and traces of functions of bounded variation on metric spaces"

The article is the result of the author's independent research.

1. Introduction

The purpose of this thesis is to study functions of bounded variation, abbreviated as BV functions, on metric measure spaces. In the classical Euclidean setting, BV functions are defined as integrable functions whose weak partial derivatives are signed measures of finite mass. Thus they form a more general class of functions than Sobolev functions, whose weak partial derivatives are integrable functions. When studying p-integrable functions in the "geometric" case $p = 1$, and in the study of the calculus of variations, BV functions are often a natural class to work with.

On the real line, BV functions can be defined in a natural way, by essentially demanding the oscillation of a function to be finite. A central problem in the history of the field has been to find a suitable generalization of this definition for several variables, see [7, Section 3.12]. This problem was eventually solved by De Giorgi and others, and several equivalent definitions of BV functions on Euclidean spaces are now available. One of them is formulated by means of weak partial derivatives as above, while another one, more suitable for generalization to metric measure spaces, involves approximation of a function by Lipschitz functions.

Accordingly, on a metric measure space (X, d, μ) equipped with a metric d and a Radon measure μ, we define BV functions through a relaxation procedure as follows. First we define upper gradients, which are a generalization of the Euclidean gradient, or more precisely its modulus. Following Miranda [60], we then define the total variation $\|Du\|(X)$ for an integrable function $u \in L^1(X)$ by

$$\|Du\|(X) := \inf \left\{ \liminf_{i \to \infty} \int_X g_i \, d\mu : \, u_i \in \mathrm{Lip}_{\mathrm{loc}}(X), \, u_i \to u \text{ in } L^1_{\mathrm{loc}}(X) \right\},$$

where each g_i is an upper gradient of u_i. Now $u \in \mathrm{BV}(X)$ if $\|Du\|(X) < \infty$.

One of the main purposes of this thesis is to study other characterizations of BV functions. To begin with, we note that since BV functions are defined by a relaxation procedure by means of Newton-Sobolev, or Newtonian functions $N^{1,1}(X)$, which are integrable functions with integrable upper gradients, it is expected that the various characterizations of these two classes are often quite analogous. Characterizations of Newtonian functions are presently quite well understood, but characterizations of BV functions as well as the relationship between the two classes less so. Toward a better understanding of this relationship, in this thesis we characterize those BV functions that are in fact Newtonian functions.

In the general metric setting, it is previously known that BV functions can be characterized by a Poincaré-type inequality involving a Radon measure of finite mass, as shown by Miranda [60]. From this, other characterizations can be derived. In particular, we prove a pointwise characterization of BV functions, involving the maximal function of a Radon measure of finite mass. This type of characterization is previously known for Newtonian functions, but for BV functions it is new also in the Euclidean setting.

While the metric measure space (X, d, μ) we work with is far more general than the classical Euclidean space $\mathbb{R}^n$, we do make certain assumptions on the space. First, we assume that the measure μ is doubling, meaning that doubling the radius of a ball increases its measure at most by a given factor. Second, in most of the thesis we assume that the space supports a Poincaré inequality, which gives control over the oscillation of a function by means of its upper gradient. These two assumptions are standard in recent literature on analysis on metric measure spaces.

An important special case of BV functions are sets of finite perimeter, which are sets $E \subset X$ whose characteristic functions are in the class $\mathrm{BV}(X)$. For such sets, the most important classical characterization, first proved by Federer [23], is that a set is of finite perimeter if and only if the codimension one Hausdorff measure of the set's measure theoretic boundary is finite. In the general metric setting, one direction of this equivalence has been established by Ambrosio [4]. In this thesis, we prove a partial result in the other direction by means of so-called relative isoperimetric inequalities.

Moreover, we also take another approach to proving the Federer-type characterization, by first noting that the classical proof strongly depends on the properties of parallel lines on a Euclidean space. We show that the proof can be extended to the more general metric setting by assuming the existence of a suitable curve family, known as a Semmes family of curves, between every pair of points in the space. Crucially, the relevant uniformity properties of parallel Euclidean lines can be distilled into a few conditions that can be expressed for curves in the general metric setting.

In this thesis we also consider a generalization of BV functions that has previously been studied in the Euclidean setting, given by functionals of linear growth. This type of functionals are similar to the total variation functional, with the difference that in their definition the upper gradient g_i is replaced by $f(g_i)$, where f is a real-valued function satisfying certain conditions. We study an integral representation for functionals of linear growth by means of the variation measure, but contrary to the Euclidean case, the functional and the integral representation are only comparable instead of being equal.

By applying the integral representation to a minimization problem related to the functional of linear growth, we are led to the study of various fine properties of BV functions. Since the minimization problem involves boundary values, the concept of boundary traces of BV functions becomes relevant. These are well understood in Euclidean spaces, but appear not to have been studied in the general metric setting. We present two ap-

proaches to constructing boundary traces. Both also involve the concept of extensions of BV functions from a domain to the whole space, motivating us to establish certain sufficient conditions for the existence of such extensions. Finally, our analysis of traces produces some novel pointwise results on the behavior of a BV function in its jump set.

This thesis is organized as follows. In Sections 2 to 8 we first introduce the relevant definitions, previously known results, and some historical background, and proceed to present the main results of our study, as well as various unifying ideas and techniques behind them. The last part of the thesis contains the five original articles.

2. Analysis on metric measure spaces and BV functions

In the past two decades, Sobolev functions, functions of bounded variation, various types of minimization problems etc. have been defined and studied in the setting of general metric measure spaces. While these topics have been more thoroughly studied in the more specific contexts of Euclidean spaces, Riemannian manifolds, Carnot-Carathéodory spaces, etc., it has turned out that many of the relevant concepts and results can be established with rather few assumptions on the space. By a metric measure space (X, d, μ) we mean a complete, separable metric space X equipped with a metric d and a Radon measure μ — even the completeness is superfluous for many results. Later we will often refer to this type of space as the "general metric setting", or simply as a "metric space", usually with the additional assumptions of a doubling measure and a Poincaré inequality, which will be defined in this section. On the other hand, by a "Euclidean space" or the "Euclidean setting" we mean, unless otherwise specified, the standard space $\mathbb{R}^n$ with the Euclidean distance and (unweighted) Lebesgue measure. Sometimes we use the word "classical" when referring to Euclidean results.

One of the central problems when introducing analysis on metric spaces is finding suitable definitions for Sobolev functions and "first order calculus". As in most of the relevant literature, in this thesis we assume all functions to be (extended) real-valued. One possible extension of Sobolev functions to metric spaces was presented by Hajłasz in [29]. For $1 \leq p \leq \infty$, a function $u \in L^p(X)$ belongs to the Hajłasz-Sobolev space $M^{1,p}(X)$ if there is a function $g \in L^p(X)$, called the Hajłasz gradient, such that

$$|u(x) - u(y)| \leq d(x,y)\big(g(x) + g(y)\big)$$

for μ-almost every $x, y \in X$, that is, for all x, y outside a set of μ-measure zero. For $1 < p \leq \infty$, this type of pointwise inequality is satisfied by the classical Sobolev functions $u \in W^{1,p}(\mathbb{R}^n)$, and in fact the inequality gives a characterization as long as $u \in L^p(\mathbb{R}^n)$. A counterexample in the case $p = 1$ is given in [28, Example 3]; unfortunately, this is precisely the relevant case in the study of BV functions.

Another property, and in a sense drawback, of Hajłasz-Sobolev functions is that the Hajłasz gradient is determined by the global behavior of the function, while usually any generalization of a derivative is expected to be a local concept. Moreover, Hajłasz-Sobolev functions automatically

satisfy a Poincaré inequality regardless of the structure of the underlying metric space, demonstrating how restricted the class is. Another possible extension of Sobolev functions to metric spaces was given by Heinonen and Koskela in [41], where so-called *upper gradients* (originally termed very weak gradients) were introduced. A nonnegative Borel function g is an upper gradient of an extended real-valued function u if for every curve γ with end points x and y — we always assume a curve to be rectifiable and parametrized by arc-length — we have

$$|u(x) - u(y)| \leq \int_{\gamma} g\, ds. \tag{2.1}$$

Now we are ready to define the class of Newton-Sobolev, or Newtonian functions, as given by Shanmugalingam in [65]. For $1 \leq p < \infty$, the Newtonian space $N^{1,p}(X)$ consists of functions $u \in L^p(X)$ that have an upper gradient $g \in L^p(X)$. On the Euclidean space $\mathbb{R}^n$, we have $W^{1,p}(\mathbb{R}^n) = N^{1,p}(\mathbb{R}^n)$ for all $1 \leq p < \infty$, with suitable choice of representatives. As the case $p = 1$ is included here, Newtonian functions can be considered a natural extension of Sobolev functions to metric spaces, and the class is employed in this thesis as well as in plenty of recent literature.

One disadvantage of upper gradients is that they are not preserved under L^p-convergence. This is one motivation for the introduction of *p-weak upper gradients*, first defined in [52]. For the definition, we first need the concept of *p-modulus* of a curve family, as given in [25] or in [61], and later defined in the metric setting in [65]. If Γ is a subset of all curves on the space, we define the p-modulus of Γ as

$$\mathrm{Mod}_p(\Gamma) := \inf \int_X \rho^p\, d\mu,$$

where the infimum is taken over all nonnegative Borel functions ρ on X that satisfy $\int_{\gamma} \rho\, ds \geq 1$ for every $\gamma \in \Gamma$. If a property holds for all curves apart from a family Γ with $\mathrm{Mod}_p(\Gamma) = 0$, we say that the property holds for p-almost every curve. Now we can give the following definition: a nonnegative μ-measurable function g is a p-weak upper gradient of an extended real-valued function u if (2.1) holds for p-almost every curve γ on the space. It can be shown that p-weak upper gradients are preserved under L^p-convergence [13, Proposition 2.2]. Moreover, from the definition of p-modulus it follows quite easily that every p-weak upper gradient can be approximated in the L^p-norm by upper gradients. Another advantage of p-weak upper gradients is that for every $u \in N^{1,p}(X)$, there is a minimal p-weak upper gradient, denoted by $g_u \in L^p(X)$, that satisfies $g_u(x) \leq g(x)$ for μ-almost every $x \in X$, for any other p-weak upper gradient $g \in L^p(X)$ [30, Theorem 7.16].

Based on the definition of p-modulus, it is straightforward to show that a Newtonian function $u \in N^{1,p}(X)$ is absolutely continuous on p-almost every curve. This is analogous with the fact that Sobolev functions on Euclidean spaces are absolutely continuous on almost every line parallel to a coordinate axis, see e.g. [22, p. 164].

Yet another definition of Sobolev functions on metric spaces is given by a relaxation procedure in [20]. In this definition, one considers functions $u \in L^p(X)$ for which there is a sequence $u_i \to u$ in $L^p(X)$ such that

a sequence of upper gradients g_i of u_i is bounded in $L^p(X)$. This definition turns out to be equivalent with the definition of Newtonian functions when $1 < p < \infty$, but in the case $p = 1$ the definition gives functions of bounded variation instead. More precisely, given a function $u \in L^1_{\text{loc}}(X)$, we define the *total variation* of u as

$$\|Du\|(X) := \inf \left\{ \liminf_{i \to \infty} \int_X g_i \, d\mu : u_i \in \text{Lip}_{\text{loc}}(X), \, u_i \to u \text{ in } L^1_{\text{loc}}(X) \right\},$$
(2.2)

where each g_i is an upper gradient — or 1-weak upper gradient — of u_i. We say that a function $u \in L^1(X)$ is a function of bounded variation, $u \in \text{BV}(X)$, if $\|Du\|(X) < \infty$. When $u = \chi_E$, we say that the set E is of *finite perimeter*. Often we also use the more convenient notation $P(E, X) := \|D\chi_E\|(X)$. The total variation can also be defined in any open set Ω, instead of the whole space X. The quantity $\|Du\|(A)$ can then be defined for any set A by approximation:

$$\|Du\|(A) := \inf\{\|Du\|(\Omega) : \Omega \text{ open}, A \subset \Omega\}.$$

Miranda has shown that by this definition, $\|Du\|$ is a Radon measure, called the variation measure [60, Theorem 3.4]. On the other hand, in this thesis we use the fact that *Newtonian* functions can be defined in any μ-measurable set $A \subset X$ by considering it as a metric space in its own right. Of course, we could define $\text{BV}(A)$ as well, but we prefer to use the above procedure of approximating general sets by open sets in order to achieve the measure property for $\|Du\|$.

For the possible definitions of BV functions on Euclidean spaces, we refer to the beginning of Section 5. Classical treatments of BV functions can be found in [23], [7], [22], [27], [67], [11], and more recently [56].

As indicated above, the space $\text{BV}(X)$ is in some ways a natural limit of the Newtonian spaces as $p \to 1^+$. Of course, Newtonian spaces $N^{1,p}(X)$ are also defined for $p = 1$, but the space $N^{1,1}(X)$ lacks some of the good properties of the space $\text{BV}(X)$, such as a compactness result, see [60, Theorem 3.7]. Newtonian functions and BV functions are also otherwise closely connected — for example, the theory of BV functions is useful in the study of Newtonian functions, especially in the case $p = 1$, when the L^p-boundedness of the Hardy-Littlewood maximal operator is not available. See for example the study of Lebesgue points in [49].

A standing assumption on the space which we maintain throughout the thesis, and which is also standard in recent literature, is that the measure μ is *doubling*. This means that for some constant $c_d \geq 1$, known as the doubling constant of the measure, and every ball $B = B(x, r) := \{y \in X : d(y, x) < r\}$, we have

$$0 < \mu(B(x, 2r)) \leq c_d \mu(B(x, r)) < \infty.$$

Note that we will often write λB for $B(x, \lambda r)$. On a metric space, a ball B does not necessarily have a unique center point and radius, but we assume every ball to come with a prescribed center and radius. By iterating the doubling condition, we easily get

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq C \left(\frac{r}{R} \right)^Q$$
(2.3)

for every $0 < r \le R$ and $y \in B(x, R)$, and some $Q > 1$ and $C > 0$ that only depend on c_d. In this thesis we follow the standard practice of denoting by C a positive constant whose value and dependence on other constants we do not always specify.

The doubling condition ensures that the Vitali covering theorem holds, allowing to establish various classical results, such as the Radon-Nikodym differentiation theorem, on the metric space — see e.g. [42]. It is also a useful fact that a complete metric space equipped with a doubling measure is proper, that is, closed and bounded sets are compact. Note that the Lebesgue measure on a Euclidean space is obviously doubling, and for other examples of spaces carrying doubling measures, see e.g. [9] or [66].

As mentioned, in this thesis we always assume the measure μ to be doubling, but it is still important for us to note that many results hold without this assumption. This is because the measure μ might not be doubling when we consider a set $D \subset X$ as a metric space in its own right, and *restrict* the measure μ to D. For example, in Theorem 4.6 of Publication IV we show that a particular function defined on the whole space is in the class $N^{1,1}(D)$ for a given set $D \subset X$. Despite the fact that the measure μ is not necessarily doubling in D, in our proof we are still able to apply certain results on Newtonian functions to the class $N^{1,1}(D)$.

Another standard assumption in analysis on metric spaces is the Poincaré inequality. It is obvious that upper gradients do not give much control of a function unless there are plenty of curves (assumed to be rectifiable) on the space. For $1 \le q, p < \infty$, we say that the space X supports a (q, p)-Poincaré inequality if for all locally integrable functions u on X and all p-weak upper gradients g of u, all balls $B = B(x, r)$, and some constants $c_P > 0$, $\lambda \ge 1$, we have

$$\left(\fint_B |u - u_B|^q \, d\mu \right)^{1/q} \le c_P r \left(\fint_{\lambda B} g^p \, d\mu \right)^{1/p},$$

where

$$u_B := \fint_B u \, d\mu := \frac{1}{\mu(B)} \int_B u \, d\mu.$$

Essentially the Poincaré inequality enables us to control the oscillation of a function u by means of its upper gradient g. With BV functions, we are mostly interested in the case $p = 1$ and $q = 1$; however, it is sometimes a useful fact that by the Sobolev embedding theorem, the number q can be increased to the value $Q/(Q-1)$, where Q was defined in (2.3). By applying the $(q, 1)$-Poincaré inequality to a sequence of approximating functions in the definition of the total variation, we get a BV version of the inequality:

$$\left(\fint_B |u - u_B|^q \, d\mu \right)^{1/q} \le c_P r \frac{\|Du\|(\lambda B)}{\mu(\lambda B)}.$$

For a list of spaces that support a Poincaré inequality, see e.g. Chapter 12 of [30]. Examples include Euclidean spaces, compact Riemannian manifolds, complete Riemannian manifolds with nonnegative Ricci curvature (see [63, Section 3.3.5]), Carnot-Carathéodory spaces, and certain topological manifolds (see [64] as well as Section 6 of this thesis). In general,

support of a Poincaré inequality on a space can be analyzed through various necessary and sufficient conditions, many of which we consider in this thesis. The Poincaré inequality as well as the Newtonian spaces $N^{1,p}(X)$ and the doubling property of the measure extend in a natural way from X to its completion [2, Proposition 7.1]. This means that assuming X to be complete in the first place, as we do in this thesis, is not very restrictive.

In nearly all of this thesis, the Poincaré inequality is a standing assumption. On a few occasions it is used explicitly, for example when we construct discrete convolutions of BV functions, see Section 4. On several occasions we use a well-known consequence of the Poincaré inequality, namely quasiconvexity of the space, see e.g. [31, Proposition 4.4]. Quasiconvexity means that there is a constant $L \geq 1$ such that every pair of points $x, y \in X$ can be connected by a curve whose length is at most $Ld(x,y)$. This implies that there is a bi-Lipschitz change of the metric that results in a *geodesic* space, where the distance between any two points is precisely the length of the shortest curve connecting them.

Bi-Lipschitz transformations of the metric are highly convenient since many interesting properties such as the doubling property of the measure and any Poincaré inequality are preserved in such transformations, even though the constants may change. For example, in Publication I we wish to prove a Poincaré-type inequality starting from a pointwise inequality. However, we need an auxiliary geometric lemma which we are only able to prove in the case of a geodesic space. Thus we employ a bi-Lipschitz change of the metric resulting in a geodesic space, and at the end of the proof we switch back to the original space. In Publication II we need the continuity of the measure $\mu(B(x,r))$ both as a function of the point x and of the radius r. This type of continuity holds on a geodesic space due to the so-called annular decay property shown by Buckley [18, Corollary 2.2], so again we perform a change of the metric.

Note that in some of the literature on BV functions on metric spaces, and also in Publications I and II of this thesis, the upper gradient g_i in the definition of the total variation (2.2) is replaced by the local Lipschitz constant $\operatorname{lip} u_i$, which is defined as

$$\operatorname{lip} u(x) := \liminf_{r \to 0} \sup_{y \in B(x,r)} \frac{|u(y) - u(x)|}{d(y,x)}.$$

The local Lipschitz constant is known to be an upper gradient of a locally Lipschitz function, see e.g. [20, Proposition 1.11]. Conversely, if $1 \leq p < \infty$ and the space supports a $(1,p)$-Poincaré inequality, we have for any $u \in \operatorname{Lip}_{\operatorname{loc}}(X)$ that $\operatorname{lip} u(x) \leq Cg_u(x)$ for μ-almost every $x \in X$, where g_u is the minimal p-weak upper gradient of u [20, Theorem 4.26]. Here the constant C depends only on the doubling constant and the constants in the Poincaré inequality. This implies that the two definitions give comparable values of the total variation, and thus produce the same space $\operatorname{BV}(X)$, as long as the space X supports a $(1,1)$-Poincaré inequality. For $1 < p < \infty$, we have in fact $\operatorname{lip} u(x) = g_u(x)$ for μ-almost every $x \in X$ [20, Theorem 6.1].

A $(1,p)$-Poincaré inequality also implies that Lipschitz functions are dense in $N^{1,p}(X)$ for any $1 \leq p < \infty$ [13, Theorem 5.47]. Thus on spaces

that support a $(1,1)$-Poincaré inequality, we have by the definition of the total variation that $N^{1,1}(X) \subset \mathrm{BV}(X)$, an inclusion that holds on Euclidean spaces almost by definition, see Section 5. On the other hand, in Publication II we do not always assume the space to support a Poincaré inequality, and then it seems more natural to use the local Lipschitz constant in the definition of the total variation, because on some spaces with few curves, the zero function is an upper gradient of every function. However, it has been recently shown in [6, Theorem 1.1] that the definitions are equivalent and give the same value of the total variation even without the assumptions of a Poincaré inequality or a doubling measure, and likewise we still have $N^{1,1}(X) \subset \mathrm{BV}(X)$.

3. Coarea formulas and relative isoperimetric inequalities

In classical measure theory, a central result is the so-called coarea formula in which an n-dimensional integral is presented as a double integral of dimensions $n-k$ and k, generalizing Fubini's theorem, see e.g. [23, Section 3.2] or [7, Theorem 2.93]. See also [57] for some generalizations to the general metric setting. For BV functions on metric spaces, there exists a coarea formula that is likewise of central importance, and it reads as follows: given a locally integrable function u on an open set Ω, we have

$$\|Du\|(\Omega) = \int_{-\infty}^{\infty} P(\{u > t\}, \Omega)\,dt, \tag{3.1}$$

where we use the abbreviation $\{u > t\} := \{x \in X : u(x) > t\}$. This was proved in [60, Proposition 4.2], following the Euclidean version of the proof given in [22, p. 185]. The coarea formula states that we can present the total variation of u as an integral of the perimeters of the super-level sets. These sets are necessarily of finite perimeter for almost every $t \in \mathbb{R}$, provided that $\|Du\|(\Omega) < \infty$. The coarea formula plays a major role in the proofs of several key results in BV theory. This is natural, since sets of finite perimeter are a special case of BV functions and allow for several strong results concerning for example representation of the perimeter measure, see [4, Theorem 5.3]. The coarea formula can then be applied to extend these results to general BV functions — see e.g. the decomposition of the variation measure presented in [9, Theorem 5.3].

Another important tool in analysis of BV functions on metric spaces is the *relative isoperimetric inequality*. This is simply the BV version of the Poincaré inequality, in the special case where u is the characteristic function of a μ-measurable set E:

$$\min\{\mu(B \cap E), \mu(B \setminus E)\} \le c_I r \|D\chi_E\|(\lambda B)$$

for any ball $B = B(x, r)$, some constant $c_I > 0$, and the same dilation factor $\lambda \ge 1$ as in the Poincaré inequality. Note that the left-hand side is comparable to $\int_B |\chi_E - (\chi_E)_B|\,d\mu$.

The relative isoperimetric inequality is, in fact, equivalent with the $(1, 1)$-Poincaré inequality. As noted above, the Poincaré inequality implies the relative isoperimetric inequality, so let us consider the converse. According to the characterizations of Poincaré inequalities given in [44, Theorem 2], it is enough to consider a Lipschitz function u with compact

support on X. Denote the super-level sets of u by $E_t := \{u > t\}$, $t \in \mathbb{R}$, and let $B = B(x, r)$ be a ball. Then we estimate, using the coarea formula for BV functions,

$$
\begin{aligned}
r \int_{\lambda B} g_u \, d\mu \geq r \|Du\|(\lambda B) &= r \int_{-\infty}^{\infty} \|D\chi_{E_t}\|(\lambda B) \, dt \\
&\geq c_I^{-1} \int_{-\infty}^{\infty} \min\{\mu(B \cap E_t), \mu(B \setminus E_t)\} \, dt \\
&\geq 2^{-1} c_I^{-1} \int_{-\infty}^{\infty} \int_B |\chi_{E_t} - (\chi_{E_t})_B| \, d\mu \, dt.
\end{aligned}
$$

Applying simple computations on the last line, we are able to recover the quantity $|u - u_B|$ by integrating the characteristic functions χ_{E_t} with respect to t. In these computations, the assumption that u is a Lipschitz function is helpful. Thus we obtain the $(1, 1)$-Poincaré inequality.

In Publication II, we examine equivalence between the Poincaré inequality, the relative isoperimetric inequality, and various *strong relative isoperimetric inequalities*. The latter are similar to the relative isoperimetric inequality, but on the right-hand side the perimeter is replaced by the codimension one Minkowski content or Hausdorff measure of either the topological boundary or the measure theoretic boundary of the set E. For example, the inequality can read

$$
\min\{\mu(B \cap E), \mu(B \setminus E)\} \leq c_S r \mu^+(\lambda B \cap \partial E) \tag{3.2}
$$

or

$$
\min\{\mu(B \cap E), \mu(B \setminus E)\} \leq c_S r \mathcal{H}(\lambda B \cap \partial E), \tag{3.3}
$$

for any μ-measurable set E, any ball $B = B(x, r)$, and constants $c_S > 0$ and $\lambda \geq 1$. Here the codimension one Minkowski content μ^+ is defined as

$$
\mu^+(A) := \liminf_{h \to 0} \frac{\mu\left(\bigcup_{x \in A} B(x, h)\right)}{2h} \quad \text{for } A \subset X,
$$

whereas the codimension one Hausdorff measure is given by the limit

$$
\mathcal{H}(A) := \lim_{R \to 0} \mathcal{H}_R(A),
$$

where the restricted Hausdorff content $\mathcal{H}_R$ is defined as

$$
\mathcal{H}_R(A) := \inf\left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} \; : \; A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), \; r_i \leq R \right\}.
$$

Strong relative isoperimetric inequalities have been previously defined in the metric setting in [17] and [48]. In particular, in [48] a suitable strong relative isoperimetric inequality was assumed to hold, and the inequality was then used to prove sufficient conditions for finite perimeter and finite variation. Motivated by this, we wish to examine whether such an inequality can be derived from a (q, p)-Poincaré inequality.

The above understood, we choose to limit ourselves to the case $p = 1$, which is the relevant case in the study of BV functions, and in which geometric quantities such as Hausdorff measures and Minkowski contents naturally arise. The case $p > 1$ has been studied in [59] and [46], where

equivalence between the $(1, p)$-Poincaré inequality and various *isocapacitary inequalities* was investigated, by use of coarea formulas and other methods similar to ours. On the other hand, in Publication II the parameter q of the (q, p)-Poincaré inequality is allowed to take any value in the interval $[1, \infty)$, but here we restrict our presentation to the case $q = 1$ for simplicity.

In the Euclidean setting, the equivalence between isoperimetric inequalities and Poincaré or Sobolev inequalities has been studied in e.g. [58] and [59] by Maz'ya, whereas certain aspects of the general metric case have been studied in [17]. Usually the same basic components are included in the proof establishing the implication between any of these inequalities. If we want to prove a Poincaré inequality starting from a strong relative isoperimetric inequality, we can use the strategy described earlier in this section. However, the type of coarea formula required depends on the type of strong relative isoperimetric inequality at our disposal. For example, in Publication II we prove and make use of a coarea inequality of the form

$$\int_B \operatorname{Lip} u \, d\mu \geq \int_{-\infty}^{\infty} \mu^+(B \cap \partial\{u > t\}) \, dt,$$

where $u \in \operatorname{Lip}(X)$ and $\operatorname{Lip} u$ is a local Lipschitz constant. This type of inequality is originally from [17, Lemma 3.1], and it is tailor-made for use with the strong relative isoperimetric inequality (3.2).

In Publication II we prove the somewhat unexpected result that inequalities (3.2) and (3.3) as well as the $(1, 1)$-Poincaré inequality are quantitatively equivalent, even though it is well-known that the codimension one Minkowski content μ^+ and Hausdorff measure $\mathcal{H}$ are not, in general, comparable. Since the Poincaré inequality is a standard assumption in analysis on metric spaces, the most interesting question here is how to get from a Poincaré inequality to a strong relative isoperimetric inequality. In particular, we are interested in the following inequality that is stronger than (3.2) and (3.3), and originally presented in [48]:

$$\min\{\mu(B \cap E), \mu(B \setminus E)\} \leq c_S r \mathcal{H}(\lambda B \cap \partial^* E)$$

for any ball $B = B(x, r)$, and constants $c_S > 0$ and $\lambda \geq 1$. Crucially, on the right-hand side we now have, instead of the topological boundary ∂E, the *measure theoretic boundary* $\partial^* E$. This is defined as follows:

$$\partial^* E := \left\{ x \in X : \limsup_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0, \quad \limsup_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0 \right\}.$$

In [17, Theorem 1.1] it was shown that a Poincaré inequality implies a strong relative isoperimetric inequality, but the latter was essentially the weaker inequality (3.2). In this case the proof consisted of approximating the characteristic function χ_E with Lipschitz functions to which the Poincaré inequality could be applied.

When proving an inequality with the Hausdorff measure on the right-hand side, we can use the same type of approach, but the details are quite different, since we need to approximate the function χ_E with Newtonian

functions instead of Lipschitz functions. We use a method first presented in [14], where a function with a prescribed upper gradient is constructed. Fix a μ-measurable set E and a ball $B = B(x, r)$, and let λ be the dilation factor from the $(1, 1)$-Poincaré inequality that we assume to hold. Assuming that $\mathcal{H}(\lambda \overline{B} \cap \partial E) < \infty$, for $\delta > 0$ we pick a cover of $\lambda \overline{B} \cap \partial E$ with balls $\{B_i\}_{i=1}^{\infty}$ such that $r_i < \delta$ and

$$\sum_{i=1}^{\infty} \frac{\mu(B_i)}{r_i} < \mathcal{H}(\lambda \overline{B} \cap \partial E) + \delta.$$

Define the Borel function

$$g := \sum_{i=1}^{\infty} \frac{\chi_{2B_i}}{r_i} + \infty \chi_{X \setminus \lambda \overline{B}}.$$

Then define

$$u(x) := \min \left\{ 1, \inf_{\gamma} \int_{\gamma} g \, ds \right\},$$

where the infimum is taken over curves connecting x to $(B \setminus E) \setminus \bigcup_{i=1}^{\infty} 2B_i$. Now g is, essentially by definition, an upper gradient of u, and its L^1-norm in the ball λB is at most $c_d(\mathcal{H}(\lambda \overline{B} \cap \partial E) + \delta)$. It is straightforward to check that the function u converges to χ_E in $L^1(B)$ as $\delta \to 0$. By applying the $(1, 1)$-Poincaré inequality to the functions u and g, we get the result.

The remaining problem is that we have worked with the whole boundary ∂E instead of the measure theoretic boundary $\partial^* E$. By a little further analysis, as presented in Publication II, we can conclude that we could carry out the above proof with the measure theoretic boundary, if we could show that 1-almost every curve that travels from the measure theoretic interior I of the set E to the measure theoretic exterior O must intersect the measure theoretic boundary. The sets I and O are defined as

$$I := \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} = 0 \right\}$$

and

$$O := \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} = 0 \right\}.$$

In the Euclidean setting, when proving the Federer-type characterization of sets of finite perimeter, establishing this type of result for curves is the crux of the proof, see [23, Section 4.5] or [22, p. 222]. However, in the Euclidean case it is enough to consider lines parallel to coordinate axes. In the general metric setting, the good uniformity properties of parallel lines are not available. To compensate for this, in Publication II we define an *extended measure theoretic boundary* as follows:

$$\partial_1^* E := \left\{ x \in X : \limsup_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))r} > 0, \quad \limsup_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))r} > 0 \right\}.$$

Due to the additional factor r in the denominator, this is clearly a bigger set than the (ordinary) measure theoretic boundary. Now we are able to show that 1-almost every curve travelling from the measure theoretic interior of E to the measure theoretic exterior intersects the extended

measure theoretic boundary. By this and the above discussion, we are able to establish a slightly weakened version of the desired strong relative isoperimetric inequality, as follows:

$$\min\{\mu(B \cap E), \mu(B \setminus E)\} \leq c_S r \mathcal{H}(\lambda B \cap \partial_1^* E).$$

In the following two sections, we discuss how this and the other relative isoperimetric inequalities can be used to analyze BV functions.

4. Discrete convolutions

In analysis on metric spaces, an essential tool that is also used in proving many of the results of this thesis is the *discrete convolution*. In classical analysis on Euclidean spaces, one can approximate a function by a convolution, which for two functions u and ϕ is given by

$$\phi * u(x) := \int_{\mathbb{R}^n} \phi(y) u(x - y)\, dy, \qquad x \in \mathbb{R}^n.$$

Similarly, one can compute the convolution of a measure ν and a function ϕ. Often the function ϕ is taken to be a *mollifier* ϕ_δ. This means that ϕ_δ, with $\delta > 0$, is a smooth function with compact support in $\overline{B}(0, \delta)$, and $\|\phi_\delta\|_{L^1(\mathbb{R}^n)} = 1$. The convolutions $\phi_\delta * u$ of a function $u \in \mathrm{BV}(\mathbb{R}^n)$ have very nice properties: they are smooth functions that converge in $L^1(\mathbb{R}^n)$ to u as $\delta \to 0$, and their gradients can be computed simply by mollifying the variation measure, i.e. $\nabla(\phi_\delta * u) = \phi_\delta * Du$. Note that in the Euclidean setting, the variation measure is vector-valued — see also Section 5.

In the general metric setting we do not have such results. However, a good substitute for convolutions are so-called discrete convolutions. To define them, we first need suitable coverings of the space, or of an open set. For the whole space, we can take balls with uniform radii, but for an open set, we need to construct a *Whitney covering*. Such coverings were already employed in [21] and [55], but for recent, careful constructions, see [15, Theorem 3.1] or [13, Theorem 3.22]. Before describing the covering, let us point out that Whitney coverings can be used to prove several important results in analysis on metric spaces, for example Gehring's lemma [54] and various extension results, see Section 8. Extension of functions by means of Whitney coverings is also used in proving a very important self-improving property of the Poincaré inequality, see [45].

Given an open set $\Omega \subset X$ and numbers $i \in \mathbb{N}$ and $\tau \geq 1$, one can construct a Whitney covering $\{B_j^i = B(x_j^i, r_j^i)\}_{j=1}^{\infty}$ with the following properties. The balls B_j^i, $j = 1, 2, \ldots$ cover Ω, the radii satisfy $r_j^i \leq 1/i$, and the dilated balls τB_j^i are contained in Ω. Moreover, if τB_j^i meets τB_k^i, then $r_j^i \leq 2 r_k^i$, and each dilated ball τB_k^i meets at most C balls τB_j^i, where the constant C only depends on the doubling constant and τ. Typically one takes τ to be a multiple of the dilation factor λ of the Poincaré inequality, say $\tau = 5\lambda$, and we assume this in the following.

Next, we construct a *partition of unity* subordinate to the Whitney covering. As shown in e.g. [15, Theorem 3.4], for every $i \in \mathbb{N}$ we can build

C/r_j^i-Lipschitz functions φ_j^i, $j = 1, 2, \ldots$, such that each of these is supported in $2B_j^i$, and

$$\sum_{j=1}^{\infty} \varphi_j^i = 1 \quad \text{on } \Omega.$$

The discrete convolutions of a locally integrable function u are then defined as

$$u_i := \sum_{j=1}^{\infty} u_{B_j^i} \varphi_j^i, \quad i \in \mathbb{N}.$$

The discrete convolutions of a function $u \in \mathrm{BV}(\Omega)$ have many good properties similar to the classical Euclidean case: they are locally Lipschitz functions that converge to u in $L^1(\Omega)$ as $i \to \infty$. Moreover, if the space supports a $(1,1)$-Poincaré inequality, every u_i has an upper gradient

$$g_i := C \sum_{j=1}^{\infty} \chi_{B_j^i} \frac{\|Du\|(5\lambda B_j^i)}{\mu(B_j^i)}, \tag{4.1}$$

where the constant C only depends on the doubling constant and the constants in the Poincaré inequality [39, Lemma 5.3]. By the properties of Whitney coverings, for every $i \in \mathbb{N}$ we now have

$$\int_{\Omega} g_i \, d\mu \leq C\|Du\|(\Omega).$$

In some ways the properties of discrete convolutions are, comparatively, even unexpectedly good: in [1] and [38] it is noted that maximal functions on metric spaces have better regularity properties when defined by means of discrete convolutions than by the usual definition of the Hardy-Littlewood maximal function.

On the other hand, for BV functions there is also another method of approximation by Lipschitz functions, presented in [19, Lemma 6.2.1]. Based on the definition of the total variation, this method gives a sequence of functions $u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ with $u_i \to u$ in $L^1_{\mathrm{loc}}(\Omega)$, $g_{u_i} \, d\mu \overset{*}{\rightharpoonup} d\|Du\|$ in Ω (weak* convergence of measures), and

$$\int_{\Omega} g_{u_i} \, d\mu \to \|Du\|(\Omega), \tag{4.2}$$

as $i \to \infty$. Due to the last property, we refer to this method as approximation by a *minimizing sequence*. Indeed, the advantage of this method is that the sequence "yields" the precise value of the total variation $\|Du\|(\Omega)$, with no constant $C > 1$ involved. Nevertheless, the abundant use of discrete convolutions in the study of BV functions in recent literature attests to the many advantages of the method compared to the minimizing sequence method. Let us discuss these.

First, discrete convolutions can be used to show that BV functions can be characterized by a Poincaré-type inequality of the form

$$\int_B |u - u_B| \, d\mu \leq cr\nu(\tau B) \tag{4.3}$$

for any ball $B = B(x, r)$, where ν is a Radon measure of finite mass and $c > 0$, $\tau \geq 1$ are constants. Moreover, we get $\|Du\| \leq C\nu$, with

$C = C(c, \tau, c_d)$. This was first proved in [60, Theorem 3.8] by Miranda, and the idea is to approximate a function $u \in L^1(X)$ with discrete convolutions, estimate the upper gradients as in (4.1), and then use a lower semicontinuity property — or simply the definition — of the total variation. Since the relative isoperimetric inequalities discussed in the previous section are inequalities of the type (4.3), with $u = \chi_E$, the finiteness of e.g. the quantity $\mathcal{H}(\partial E)$ implies that E is of finite perimeter. This type of reasoning was employed in [48, Theorem 4.6].

In Publication III, discrete convolutions are used in establishing an inequality involving a sequence of locally Lipschitz functions u_i approximating a BV function u in the L^1-norm. More precisely, the inequality is of the following form:

$$\limsup_{i \to \infty} \int_B R_{x,y} \, g_i \, d\mu \le C \int_{2B} R_{x,y} \, d\|Du\|,$$

where $C > 0$, each g_i is an upper gradient of u_i, $R_{x,y}$ is the sum of two Riesz kernels centered at points x and y, and B is a ball containing x and y. A possible method of proving this inequality would be to take a minimizing sequence u_i as described above, since then we would have the weak* convergence $g_i \, d\mu \overset{*}{\rightharpoonup} d\|Du\|$. However, it is not clear how to use this weak* convergence in connection with the discontinuous function $R_{x,y}$. By contrast, it turns out that when approximating u by discrete convolutions, the singularities of the Riesz kernels do not cause trouble, and we get the above inequality for μ-almost every x, y. As is usual with discrete convolutions, a constant C appears on the right-hand side.

In Publication IV we find that discrete convolutions have another useful property. When $u \in \mathrm{BV}(\Omega)$ and the variation measure is absolutely continuous with respect to the measure μ, i.e. $d\|Du\| = a \, d\mu$ with $a \in L^1(\Omega)$, it turns out that the sequence of upper gradients g_i of the discrete convolutions u_i is *equiintegrable*. The equiintegrability of a sequence of functions $g_i \in L^1(\Omega)$ is defined by two conditions. First, for any $\varepsilon > 0$ there is a μ-measurable set $A \subset \Omega$ with $\mu(A) < \infty$ such that

$$\int_{\Omega \setminus A} g_i \, d\mu < \varepsilon \quad \text{for all } i \in \mathbb{N}.$$

Second, for any $\varepsilon > 0$ there is $\delta > 0$ such that if $\widetilde{A} \subset \Omega$ is μ-measurable with $\mu(\widetilde{A}) < \delta$, then

$$\int_{\widetilde{A}} g_i \, d\mu < \varepsilon \quad \text{for all } i \in \mathbb{N}.$$

By the Dunford-Pettis theorem, see e.g. [7, Theorem 1.38], from any sequence of functions $g_i \in L^1(\Omega)$ that is equiintegrable, one can pick a subsequence that converges weakly in $L^1(\Omega)$. Then by Mazur's lemma (see e.g. [62, Theorem 3.12] or [42]), one can pick convex combinations of g_i that converge strongly in $L^1(\Omega)$. This type of reasoning has been previously applied in [24] to prove a certain self-improving property of Poincaré inequalities. In Publication IV we generalize the method somewhat: instead of assuming that $\|Du\|$ is absolutely continuous in the entire open set Ω, we show that in the set where it *is* absolutely continuous, the upper gradients of suitably constructed discrete convolutions are equiintegrable. In

Sections 5 and 7 we will describe how this equiintegrability property is used in this thesis.

In [50, Proposition 4.1] it is also shown that one can estimate the pointwise convergence of a sequence of discrete convolutions of a BV function outside a set of $\mathcal{H}$-measure zero instead of just a set of μ-measure zero. In proving a Leibniz rule for BV functions, both approximation methods are used in [50, Proposition 4.2]: one for each function in the product. Yet even here, the minimizing sequence approximation does not seem to have any real advantage, since discrete convolutions have, up to a constant C, also both the minimizing property (4.2) and the weak* convergence property, and the result in [50, Proposition 4.2] is also an inequality involving a constant.

Nevertheless, it is interesting to note that the two methods cannot be merged into one. Namely, the construction presented in Example 4.8 of Publication IV shows that on a metric space it is, in general, impossible to obtain equiintegrability for the upper gradients of a minimizing approximating sequence of a BV function with an absolutely continuous variation measure. This implies, by the Dunford-Pettis theorem, that while the sequence of measures $g_i \, d\mu$ weakly* converges to the variation measure $a \, d\mu$, no subsequence of the upper gradients g_i converges weakly in L^1. Notably, the space in our example is just an interval of the real line with the Euclidean distance and a Lebesgue measure weighted by a function that takes values between 1 and 2. By contrast, in the classical Euclidean setting, all the good approximating properties that have been discussed are possible to achieve simultaneously, since the gradients of convolutions even converge in L^1 to the weak gradient of u, when the latter is absolutely continuous.

5. Characterizations of BV functions

A central theme in this thesis are the various characterizations of BV functions on metric spaces. In the Euclidean setting there are two standard ways to define BV functions. Let Ω be an open set. According to the first definition, a function $u \in L^1(\Omega)$ is in the class $\mathrm{BV}(\Omega)$ if the total variation

$$\|Du\|(\Omega) := \sup \left\{ \int_\Omega u \operatorname{div} \phi \, dx : \phi \in C_c^1(\Omega; \mathbb{R}^n), \ |\phi| \le 1 \right\}$$

is finite. An equivalent definition states that a function $u \in L^1(\Omega)$ is in the class $\mathrm{BV}(\Omega)$ if there exist signed Radon measures of finite mass $\nu_1, \ldots, \nu_n$, such that

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} \, dx = - \int_\Omega \phi \, d\nu_i \quad \text{for all } \phi \in C_c^1(\Omega), \ i = 1, \ldots, n,$$

that is, the weak gradient $Du = \nu$ of u is an $\mathbb{R}^n$-valued measure with finite mass $\|Du\|(\Omega)$. It is then usually proved as a theorem that u can be approximated by smooth functions, see e.g. [7, Theorem 3.9]. This fact is essentially taken as the definition of BV functions on metric spaces, where directional derivatives are not available — see the definition given in Section 2.

A very important characterization of BV functions on metric spaces, from which several other characterizations can be derived, is that a Poincaré-type inequality of the form (4.3) characterizes the class BV. Note that in order to ensure that BV functions satisfy such as inequality, we need to assume that the space supports a $(1,1)$-Poincaré inequality. In [39] the authors present a more general formulation of this characterization, defining spaces of functions $u \in L^1_{\mathrm{loc}}(\Omega)$ for which

$$\|u\|_{A^{1,p}_{\tau,0}(\Omega)} := \lim_{r \to 0} \sup_{\mathcal{B} \in \mathcal{B}_{\tau,r}} \left\| \sum_{B \in \mathcal{B}} \left(r_B^{-1} \fint_B |u - u_B| \, d\mu \right) \chi_B \right\|_{L^p(\Omega)}$$

is finite for some $\tau \ge 1$; here $\mathcal{B}_{\tau,r}$ consists of all collections of balls B with radius $r_B \le r$ such that the balls τB are disjoint and contained in Ω. The authors show that for a function $u \in L^1(\Omega)$, the condition $\|u\|_{A^{1,1}_{\tau,0}(\Omega)} < \infty$ implies $u \in \mathrm{BV}(\Omega)$, whereas when $p > 1$ and $u \in L^p(\Omega)$, $\|u\|_{A^{1,p}_{\tau,0}(\Omega)} < \infty$ implies that $u \in N^{1,p}(\Omega)$. This is another demonstration of the fact that the class BV is sometimes a natural limit of the spaces $N^{1,p}(\Omega)$ as $p \to$

1^+. In Publication IV we show that if the space supports a $(1, 1)$-Poincaré inequality, then the class $N^{1,1}(\Omega)$ is, in turn, characterized by the Poincaré inequality

$$\int_B |u - u_B| \, d\mu \leq Cr \int_{\lambda B} a \, d\mu$$

for every ball $B = B(x, r) \subset \lambda B \subset \Omega$ and a given $a \in L^1(\Omega)$. This can again be proved by using discrete convolutions and equiintegrability of the upper gradients. On the other hand, BV functions whose variation measures are absolutely continuous are precisely functions that satisfy a Poincaré inequality of the above type. Thus we can identify this class with $N^{1,1}(\Omega)$, and in fact we prove in Publication IV the following, somewhat stronger result. Given $u \in \mathrm{BV}(\Omega)$ and any μ-measurable set $F \subset \Omega$ where $\|Du\|$ is absolutely continuous with respect to μ, we have $u|_F \in N^{1,1}(F)$ for some μ-representative of u.

All in all, there are very close connections between BV functions and Newtonian functions, and sometimes BV functions can even be utilized in the study of Newtonian functions. In the main result of [48], the goal is to show that a particular function is in the space $N^{1,p}(X)$, and the strategy is to first show that the function is in the space $\mathrm{BV}(X)$, and then to check that the singular part of the variation measure is zero.

On the other hand, Hajłasz has demonstrated in [28] and [30, Theorem 8.7] how to get from *pointwise* inequalities of the type $|u(x) - u(y)| \leq d(x, y)(g(x) + g(y))$ to Poincaré-type inequalities. For example, the pointwise inequality

$$|u(x) - u(y)| \leq Cd(x, y)\big((\mathcal{M}_{\sigma d(x,y)}g^p(x))^{1/p} + (\mathcal{M}_{\sigma d(x,y)}g^p(y))^{1/p}\big)$$

for μ-almost every $x, y \in X$, with $\sigma > 0$, implies a $(1, p)$-Poincaré inequality for the pair u, g, for any $1 \leq p < \infty$, see [30, Theorem 9.5]. Here $\mathcal{M}_{\sigma d(x,y)}$ denotes the restricted Hardy-Littlewood maximal function. In Publication I we show that BV functions can be characterized by a similar inequality

$$|u(x) - u(y)| \leq Cd(x, y)\big(\mathcal{M}_{\sigma d(x,y)}\nu(x) + \mathcal{M}_{\sigma d(x,y)}\nu(y)\big),$$

where ν is a Radon measure of finite mass. In the proof, we show that the above inequality implies a Poincaré-type inequality for the pair u, ν, and then apply the characterization of BV functions by Poincaré-type inequalities. Moreover, in Publication III we present a slightly different formulation and proof of this pointwise characterization of BV, relying more directly on the results of Hajłasz. Conversely, if the space supports a Poincaré inequality, pointwise inequalities of the above types can easily be proved for BV and Newtonian functions by a telescoping argument.

Thus far we have discussed characterizations of general BV functions. On the other hand, for sets of finite perimeter one has, at least in the Euclidean case, the very important Federer-type characterization that states that a set E is of finite perimeter if and only if $\mathcal{H}(\partial^* E) < \infty$. This was first proved by Federer in [23, Section 4.5], and a simplified version of the proof can be found in [22, p. 222]. The Federer-type characterization is quite surprising in the sense that it brings together, on the one hand the concept of the perimeter measure, which is a variational concept defined through

a relaxation procedure and readily useful in the calculus of variations, and on the other hand the codimension one Hausdorff measure which is a purely measure theoretic object. In fact, the perimeter measure and the codimension one Hausdorff measure restricted to the measure theoretic boundary are equal in the Euclidean case, and in the general metric setting, Ambrosio et al. have shown them to be comparable *if* the set is of finite perimeter, see [4, Theorem 5.3] and [9, Theorem 4.6].

To prove the other direction of the Federer-type characterization in the metric setting, namely that $\mathcal{H}(\partial^* E) < \infty$ implies that E is of finite perimeter, we could again use Miranda's characterization of BV functions by means of Poincaré-type inequalities. This is due to the fact that the relative isoperimetric inequalities discussed in Section 3 are Poincaré-type inequalities written for the characteristic functions of sets. Precisely speaking, we would need the strong relative isoperimetric inequality with the quantity $\mathcal{H}(\lambda B \cap \partial^* E)$ on the right-hand side.

As explained in Section 3, we attempt to derive this type of inequality from the $(1,1)$-Poincaré inequality in Publication II — however, we are only able to establish the weaker inequality with the measure theoretic boundary $\partial^* E$ replaced by the extended measure theoretic boundary $\partial_1^* E$. Nevertheless, this does give the following sufficient condition: if $\mathcal{H}(\partial_1^* E) < \infty$, then E is of finite perimeter. The BV coarea formula then provides a simple sufficient condition for general BV functions: $u \in L^1(X)$ is a BV function if

$$\int_{-\infty}^{\infty} \mathcal{H}(\partial_1^* \{u > t\}) \, dt < \infty.$$

6. Semmes family of curves

Since the Poincaré inequality enables any locally integrable function to be controlled by its upper gradient, and upper gradients are defined by means of curves, any space that supports a Poincaré inequality must in some sense contain an abundance of curves. Precise expressions of this fact can be found e.g. in [51, Lemma 3.2], [44, Theorem 2], and [41, Theorem 5.7]. In [44, Theorem 2] a converse is also established — if the space contains "enough curves", it supports a Poincaré inequality.

In Publication III we make the somewhat stronger assumption that there is a *geometric Semmes family of curves* between any pair of points in the space. Curve families of this type were first defined and constructed by Semmes in [64] with the motivation of proving a Poincaré inequality on certain topological manifolds. At first, we define the (ordinary) Semmes family of curves as follows: for any $x, y \in X$ there is a family of curves $\Gamma_{x,y}$ joining x and y and a probability measure $\alpha_{x,y}$ such that for any Borel set $A \subset X$,

$$\int_{\Gamma_{x,y}} \ell(\gamma \cap A) \, d\alpha_{x,y}(\gamma) \leq c_S \int_{A \cap B_{xy}} R_{x,y}(z) \, d\mu(z),$$

with $c_S > 0$. Here $\ell(\gamma \cap A)$ is the length of the curve γ in the set A. Moreover, $R_{x,y}$ is the sum of two Riesz kernels, centered at x and y, and $B_{xy} := B(x, \tau d(x, y))$, with $\tau > 1$. The above condition implies, at minimum, that there must be many curves in the family $\Gamma_{x,y}$ — for if there were, say, just a finite number of curves in $\Gamma_{x,y}$ and the set A was taken to be the union of their images, then the right-hand side would in most cases be zero, but the left-hand side would not. We also note in Publication III that the above condition implies that the space supports a $(1, 1)$-Poincaré inequality. In defining the *geometric* Semmes family of curves, we then assume some additional uniformity properties of the curves in $\Gamma_{x,y}$.

The main result of Publication III is that if a metric space supports a geometric Semmes family of curves, then the Federer-type characterization for sets of finite perimeter, which was discussed at the end of the previous section, holds. The proof combines many of the ideas and characterizations of BV functions introduced so far, as explained in the following.

First we consider one more characterization of BV functions, expressed by means of curves on the space. The idea is to investigate, for a function $u \in L^1(X)$, whether the functions $u \circ \gamma$ are in the class $\mathrm{BV}((0, \ell_\gamma))$ for curves γ. Here ℓ_γ is the length of γ — recall that we assume all curves

to be parametrized by arc-length. We prove that a bounded function $u \in L^1(X)$ is in $\mathrm{BV}(X)$ if and only if

$$\int_{\Gamma_{x,y}} \|D(u \circ \gamma)\|((0, \ell_\gamma)) \, d\alpha_{x,y}(\gamma) \leq C_0 \int_{B(x, \kappa d(x,y))} R_{x,y} \, d\nu \qquad (6.1)$$

for some constants $C_0, \kappa > 0$, a Radon measure of finite mass ν, and μ-almost every $x, y \in X$. In particular, $u \circ \gamma \in \mathrm{BV}((0, \ell_\gamma))$ for $\alpha_{x,y}$-almost every curve $\gamma \in \Gamma_{x,y}$. The proof of the sufficiency of this so-called Reshetnyak-type characterization is based on two facts: first, we can establish the following weak continuity at the end points: $|u(x) - u(y)| \leq \|D(u \circ \gamma)\|((0, \ell_\gamma))$ for almost every curve $\gamma \in \Gamma_{x,y}$. Second, the integral of the Riesz kernel $R_{x,y}$ on the right-hand side of (6.1) can easily be estimated by the maximal function of the measure ν, as noted already in [37]. Then we can use the pointwise characterization of BV functions that was discussed earlier — recall that this was, in turn, based on Miranda's characterization by Poincaré-type inequalities. A characterization of BV functions in terms of curves is well-known in the Euclidean case — where it is usually formulated for lines — and it is an analogue of the ACL or ACC property (absolute continuity on almost every line or curve) of Sobolev or Newtonian functions.

Now, to obtain the Federer-type characterization, let us briefly discuss how to establish (6.1) for $u = \chi_E$ and $\nu = \mathcal{H}|_{\partial^* E}$, with $\mathcal{H}(\partial^* E)$ finite. Using geometric arguments and the properties of the Semmes family of curves, we can first show that for μ-almost every $x, y \in X$, we have

$$\int_{\Gamma_{x,y}} \#(\gamma \cap \partial^* E) \, d\alpha_{x,y}(\gamma) \leq C \int_{3B_{xy}} R_{x,y} \, d\mathcal{H}^h|_{\partial^* E}.$$

Thus we see that in order to get (6.1), we simply need to show that

$$\|D(\chi_E \circ \gamma)\|((0, \ell_\gamma)) \leq \#(\gamma \cap \partial^* E)$$

for almost every curve $\gamma \in \Gamma_{x,y}$. Since we can pick any μ-representative of E, we can assume that $E = I$, that is, the measure theoretic interior of E. In one dimension, the total variation of a function can of course be controlled by its pointwise variation, so again we encounter the problem of showing that almost every curve or subcurve that travels from the measure theoretic interior of E to the measure theoretic exterior must pass through the measure theoretic boundary. This is proved by using the good uniformity properties of the geometric Semmes family of curves. In fact, curves in this family behave much like parallel lines on a Euclidean space: the curve bundle is "equally thick" everywhere, and the curves travel in the "same direction" and at a uniform "speed". This enables us to mimic the proof presented in [22], as explained in the following.

The idea is to define subfamilies of $\Gamma_{x,y}$ in which curves "jump" from the measure theoretic interior to the measure theoretic exterior, or vice versa, avoiding the measure theoretic boundary. In order to show that these curve families have $\alpha_{x,y}$-measure zero, we first show that the measure $\alpha_{x,y}$ is locally doubling with respect to a suitable metric on the space of curves $\Gamma_{x,y}$. This ensures that the Lebesgue differentiation theorem holds

on the space $\Gamma_{x,y}$. On the other hand, using the fact that the curves behave much like Euclidean lines, we are able to show that the density of the curves that "jump" is everywhere strictly less than one. By the Lebesgue differentiation theorem, the $\alpha_{x,y}$-measure of these curve families must then be zero.

The obvious example of a space supporting a geometric Semmes family of curves is a Euclidean space. As a more interesting example, we discuss Fred Gehring's bow-tie, which is defined as $X = X_+ \cup X_-$, with

$$X_+ := \{z \in \mathbb{R}^n : 0 \leq z_n \leq 1, |z_j| \leq z_n, j = 1, \ldots, n-1\},$$
$$X_- := \{z \in \mathbb{R}^n : -1 \leq z_n \leq 0, |z_j| \leq |z_n|, j = 1, \ldots, n-1\}.$$

We equip this space with the metric inherited from $\mathbb{R}^n$ and a Lebesgue measure weighted by $\omega(z) := |z|^{\alpha}$, with $\alpha > -n$. It turns out that this space supports a geometric Semmes family of curves precisely when $\alpha = 1 - n$. It is interesting to note that the space supports a $(1,1)$-Poincaré inequality precisely when $\alpha \leq 1 - n$ [13, Example A.24], demonstrating again the connection between the two concepts. In Publication III we also construct a geometric Semmes family in the first Heisenberg group, and note that by numerical calculations, all the conditions appear to be satisfied.

It can also be mentioned that a characterization of BV functions by means of curves, more precisely *all* curves on the space, was recently presented in [6]. Far fewer assumptions on the space were made in this paper, but as a drawback it is unclear how to proceed from the characterization given in this paper to the Federer-type characterization.

7. Functionals of linear growth

In Publication IV we consider functionals of linear growth, which are a generalization of the total variation functional that is used to define BV functions. First we let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing convex function that satisfies the linear growth condition

$$mt \leq f(t) \leq M(1 + t)$$

for all $t \geq 0$, with some constants $0 < m \leq M < \infty$. This also implies that f is Lipschitz with constant $L > 0$. Furthermore, we define

$$f_\infty := \sup_{t>0} \frac{f(t) - f(0)}{t} = \lim_{t \to \infty} \frac{f(t) - f(0)}{t} = \lim_{t \to \infty} \frac{f(t)}{t},$$

where the second equality follows from the convexity of f.

A model case that is good to keep in mind is the function $f(t) = \sqrt{1 + t^2}$, which appears in the definition of the classical surface area functional

$$C^1(\Omega) \ni u \mapsto \int_\Omega \sqrt{1 + |\nabla u|^2} \, dx.$$

Now we give the definition of the functional of linear growth. For an open set Ω and $u \in N^{1,1}(\Omega)$, one possibility would be to define it as $u \mapsto \int_\Omega f(g_u) \, d\mu$, where g_u is the minimal 1-weak upper gradient of u. However, we wish to define the functional for BV functions as well, so we need to use a relaxation procedure. For $u \in L^1_{\text{loc}}(\Omega)$, the functional of linear growth is defined by

$$\mathcal{F}(u, \Omega) := \inf \left\{ \liminf_{i \to \infty} \int_\Omega f(g_{u_i}) \, d\mu : u_i \in \text{Lip}_{\text{loc}}(\Omega), \, u_i \to u \text{ in } L^1_{\text{loc}}(\Omega) \right\},$$

where g_{u_i} is the minimal 1-weak upper gradient of u_i. In the case $f(t) = t$, the functional $\mathcal{F}(u, \cdot)$ is just the total variation $\|Du\|(\cdot)$. Due to the linear growth conditions on f, we always have

$$m\|Du\|(\Omega) \leq \mathcal{F}(u, \Omega) \leq M(\mu(\Omega) + \|Du\|(\Omega)).$$

Functionals of the type described above were introduced in the Euclidean setting in [5] — see also the discussion therein on results related to an earlier definition of $\mathcal{F}$ without the use of relaxation, which is suitable for Sobolev functions but not BV functions. A more thorough classical treatment of these functionals is given in [7], and other sources include

[26] and [11]. We point out that in the Euclidean case, gradients as well as the variation measure Du are vector-valued, and in the first two sources mentioned above, the function u under consideration is also taken to be vector-valued. Moreover, in the Euclidean case the function f is assumed to be *quasiconvex* instead of convex, but this property reduces to convexity in the scalar case, see [7, Proposition 5.41].

By contrast, in Publication IV, where we work in the general metric setting, we assume $u \in \mathrm{BV}(\Omega)$ to be scalar-valued, as is done in most recent theory of BV functions on metric spaces. Then the variation measure is merely the scalar measure $\|Du\|$. Despite these simplifications, working in the general metric setting produces significant complications compared to the Euclidean case, and these appear already in the weighted Euclidean setting. We will discuss these issues shortly.

The functional $\mathcal{F}(u, \cdot)$ has been previously studied in the metric setting in [34] and [35], where existence and regularity of minimizers of the functional are considered. In Publication IV we focus on the fundamental properties of the functional itself. First we define $\mathcal{F}(u, A)$ for arbitrary sets A by approximation from outside with open sets. Then we show that $\mathcal{F}(u, \cdot)$ is a Radon measure. This can be done in a similar fashion as in the Euclidean setting, see e.g. [7, Chapter 5.5], or in the case of the total variation $\|Du\|(\cdot)$ on metric spaces, see [60].

The main goal of Publication IV is the same as was in [5] in the Euclidean context: to produce an integral representation for $\mathcal{F}(u, \cdot)$. The motivation for this is that the rather indirect definition of $\mathcal{F}(u, \cdot)$ by means of relaxation makes the functional difficult to work with. Let us assume that $\mathcal{F}(u, \Omega) < \infty$, implying also $\|Du\|(\Omega) < \infty$. Denote the decomposition of the variation measure $\|Du\|$ into the absolutely continuous and singular parts with respect to the measure μ by $d\|Du\| = a\,d\mu + d\|Du\|^s$, and similarly write $\mathcal{F}(u, \cdot) = \mathcal{F}^a(u, \cdot) + \mathcal{F}^s(u, \cdot)$. Our result states that

$$\mathcal{F}^s(u, \Omega) = f_\infty \|Du\|^s(\Omega)$$

and

$$\int_\Omega f(a)\,d\mu \leq \mathcal{F}^a(u, \Omega) \leq \int_\Omega f(Ca)\,d\mu,$$

where the constant C depends only on the doubling constant and the constants in the Poincaré inequality. Proving the result for the singular part is fairly straightforward. The idea is simply that in the set where u varies "singularly", the weak upper gradients g_{u_i} of an approximating sequence u_i must be very large, and then $f(g_{u_i}) \approx f_\infty g_{u_i}$.

For the absolutely continuous part $\mathcal{F}^a(u, \cdot)$, we can restrict our analysis to an open set $G \subset \Omega$ where the singular parts $\|Du\|^s$ and $\mathcal{F}^s(u, \cdot)$ are arbitrarily small. The estimate from below can be proved by taking an approximating sequence of locally Lipschitz functions u_i that converges to u in $L^1_{\mathrm{loc}}(G)$, such that $\int_G f(g_{u_i})\,d\mu$ converges to the value of the functional $\mathcal{F}(u, G)$. By a standard compactness result, a subsequence of the sequence of measures $g_{u_i}\,d\mu$ converges weakly* to a Radon measure with finite mass ν, and furthermore we can show that this measure majorizes the variation measure $\|Du\|$. Then we can use an argument which states that by the

convexity of f, the functional

$$L^1(G) \ni a \mapsto \int_G f(a) \, d\mu \qquad (7.1)$$

is lower semicontinuous with respect to the weak* convergence of absolutely continuous measures. This gives us the result, but it is worth making a certain observation on this procedure. Namely, we only needed the weak* convergence of absolutely continuous measures, instead of weak or strong convergence in $L^1(G)$. On the other hand, the setting is asymmetric: f is convex but not necessary concave (and thus linear), and hence the functional (7.1) is lower but not upper semicontinuous.

Due to this asymmetry, we cannot use the same methods for the estimate from above for the absolutely continuous part $\mathcal{F}^a(u, \cdot)$. Ultimately we obtain an unexpected result: the estimate holds only up to a constant C that depends on the doubling constant and the constants in the Poincaré inequality. By contrast, in the Euclidean case it was possible to gain equality. Furthermore, Example 4.8 in Publication IV shows that the constant C cannot be discarded. In fact, the same example shows that for $u \in N^{1,1}(\Omega)$ and its minimal 1-weak upper gradient g_u, it is possible to have

$$\|Du\|(\Omega) < \int_\Omega g_u \, d\mu;$$

however, these are also comparable by a constant C. In a sense, this shows that the definition of the total variation — and of the functionals of linear growth — through relaxation on metric spaces is slightly less self-consistent than one might hope for. A question about this consistency was raised in [60] and later in [9]. It is notable that in our counterexample, u is even Lipschitz and the space is geodesic and Ahlfors-regular. In fact, the space is just an interval on the real line, with the Euclidean distance and a weighted Lebesgue measure where the weight takes values between 1 and 2.

Let us discuss the reason for the above complication. In the Euclidean case, the estimate from above for the absolutely continuous part can be proved simply by approximating u by mollifications $\phi_\delta * u$. As discussed in Section 4, the gradient of $\phi_\delta * u$ is just $\phi_\delta * Du$, and for the density of the absolutely continuous part of Du, denoted by ∇u, we get the (strong) L^1-convergence $\phi_\delta * \nabla u \to \nabla u$. In the metric case we would similarly need this type of L^1-convergence for the upper gradients of an approximating sequence of u, because the functional (7.1) was only lower semicontinuous with respect to weak* convergence of measures.

However, from Example 4.8 in Publication IV we know that for a function with an absolutely continuous variation measure $\|Du\|$, there may not be any approximating sequence of locally Lipschitz functions u_i whose upper gradients converge in L^1 to the density of the variation measure. Essentially the best solution available is to approximate u by discrete convolutions — these have equiintegrable upper gradients, so for a subsequence we get weak convergence in L^1, and for convex combinations we get strong convergence in L^1 by Mazur's lemma. The drawback is that as always with discrete convolutions, a constant C appears in the result.

In closing, let us mention that it is natural to ask how far the known theory of BV functions and the variation measure could be developed for the more general functionals of linear growth. It turns out that many results hold, at best, up to a constant in this more general case, and the easiest proofs are obtained by combining the corresponding results for the variation measure with the integral representation described above. For example, a coarea formula of the form (3.1) does not hold for the functional $\mathcal{F}(u, \cdot)$, because if $f(0) \neq 0$, the right-hand side is infinity as soon as $\mu(\Omega) > 0$. If we assume that $f(0) = 0$, and use the integral representation for $\mathcal{F}(u, \cdot)$ as well as the coarea formula for the variation measure, we can show that

$$\int_{-\infty}^{\infty} \mathcal{F}(\chi_{\{u>t\}}, \Omega)\, dt = f_{\infty} \|Du\|(\Omega).$$

If the right-hand side is always equal to $\mathcal{F}(u, \Omega)$, the functional $\mathcal{F}$ is just a multiple of the total variation, and f is then necessarily a linear function. Thus the coarea formula holds only in this special case. Nonetheless, when $f(0) = 0$, it is easy to see that $\mathcal{F}(u, \cdot)$ is always comparable to the variation measure, with the constants of comparison depending on the constants m and M determining the linear growth of f. Thus we do get the coarea formula and various other results up to a constant.

8. Traces and extensions of BV functions

In Publication IV we also consider a minimization problem related to the functionals of linear growth. For this, we need the concept of boundary values of BV functions. Let Ω and Ω^* be bounded open subsets of X such that $\Omega \Subset \Omega^*$, and assume that $h \in \mathrm{BV}(\Omega^*)$. We define $\mathrm{BV}_h(\Omega)$ as the space of functions $u \in \mathrm{BV}(\Omega^*)$ such that $u = h$ μ-almost everywhere in $\Omega^* \setminus \Omega$.

Then we can define the minimization problem. A function $u \in \mathrm{BV}_h(\Omega)$ is a minimizer of the functional of linear growth with the boundary values $h \in \mathrm{BV}(\Omega^*)$, if

$$\mathcal{F}(u, \Omega^*) = \inf \mathcal{F}(v, \Omega^*),$$

where the infimum is taken over all $v \in \mathrm{BV}_h(\Omega)$. In [34, Theorem 3.6] it was proved that the minimization problem always has a solution. The proof was based on a compactness result for BV functions, demonstrating why it was beneficial to define the functionals of linear growth for this wider class, instead of just $N^{1,1}(X)$.

With the help of the integral representation of the functional of linear growth presented in the previous section, we can prove the second major result of Publication IV. Namely, we establish, under certain assumptions on the space and the set Ω, equivalence between the above minimization problem and minimization of the functional

$$\mathcal{F}(u, \Omega) + f_\infty \int_{\partial\Omega} |T_\Omega u - T_{X \setminus \Omega} h| \theta_\Omega \, d\mathcal{H} \tag{8.1}$$

over all $u \in \mathrm{BV}(\Omega)$. Here $T_\Omega u$ and $T_{X \setminus \Omega} h$ are *boundary traces*, and θ_Ω is positive function that we will define later. While obeying the boundary values h, a function $u \in \mathrm{BV}_h(\Omega)$ may well "jump" across the boundary $\partial\Omega$, but the cost of this is a penalty term where we integrate the magnitude of the "jump" over the boundary $\partial\Omega$. In the above formulation, the class of functions over which we minimize is simply $\mathrm{BV}(\Omega)$, with no reference to the larger set Ω^*. This type of formulation is previously known in the Euclidean case [26, p. 584].

The concept of boundary traces of BV functions is also well-known in the Euclidean case, see e.g. [7, Theorem 3.87] or [27, Section 2], but seems not to have been studied in the general metric setting. In this thesis, we present two approaches to constructing boundary traces. In Publication IV, we set out by considering the concept of BV *extension domains*. We say that an open set $\Omega \subset X$ is a *strong* BV *extension domain* if there is

a constant $c_\Omega > 0$ such that for every $u \in \mathrm{BV}(\Omega)$, there is an extension $Eu \in \mathrm{BV}(X)$ for which

$$Eu|_\Omega = u, \quad \|Eu\|_{\mathrm{BV}(X)} \le c_\Omega \|u\|_{\mathrm{BV}(\Omega)}, \quad \text{and} \quad \|D(Eu)\|(\partial\Omega) = 0.$$

Without the last condition, we call Ω merely a BV extension domain. The BV norm is simply

$$\|u\|_{\mathrm{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|(\Omega).$$

Another concept that we will need is defined as follows: a μ-measurable set Ω satisfies the *weak measure density condition* if for $\mathcal{H}$-almost every $x \in \partial\Omega$,

$$\liminf_{r \to 0} \frac{\mu(B(x,r) \cap \Omega)}{\mu(B(x,r))} > 0.$$

At this point, let us list some key concepts and facts of BV theory on metric spaces. Ambrosio has shown in [4, Theorem 5.4] that if E is a set of finite perimeter, for $\mathcal{H}$-almost every $x \in \partial^* E$ we have

$$\gamma \le \liminf_{r \to 0} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} \le \limsup_{r \to 0} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} \le 1 - \gamma, \qquad (8.2)$$

where $\gamma \in (0, 1/2]$ only depends on the doubling constant c_d and the constants in the Poincaré inequality c_P, λ. For a set of finite perimeter $E \subset X$ and an open set $\Omega \subset X$, we know that

$$\|D\chi_E\|(\Omega) = \int_{\Omega \cap \partial^* E} \theta_E \, d\mathcal{H},$$

where $\theta_E : X \mapsto [\alpha, c_d]$ with $\alpha = \alpha(c_d, c_P, \lambda) > 0$, see [4, Theorem 5.3], [9, Theorem 4.6]. This means that the perimeter measure and the codimension one Hausdorff measure restricted to the measure theoretic boundary are comparable.

The lower and upper approximate limits of a μ-measurable function u are defined as

$$u^\wedge(x) := \sup\left\{ t \in \overline{\mathbb{R}} : \lim_{r \to 0} \frac{\mu(B(x,r) \cap \{u < t\})}{\mu(B(x,r))} = 0 \right\}$$

and

$$u^\vee(x) := \inf\left\{ t \in \overline{\mathbb{R}} : \lim_{r \to 0} \frac{\mu(B(x,r) \cap \{u > t\})}{\mu(B(x,r))} = 0 \right\}.$$

The jump set of u is defined as

$$S_u := \{ x \in X : u^\wedge(x) < u^\vee(x) \}.$$

Outside the jump set, i.e. in $X \setminus S_u$, $\mathcal{H}$-almost every point is a Lebesgue point of a function $u \in \mathrm{BV}(X)$, as shown by Kinnunen et al. in [50, Theorem 3.5].

According to [9, Theorem 5.3], the variation measure of a function $u \in \mathrm{BV}(X)$ can be decomposed as

$$d\|Du\| = a \, d\mu + d|D^c u| + \theta_u \, d\mathcal{H}^h|_{S_u},$$

where $a \in L^1(X)$, $|D^c u|$ is the so-called Cantor part and

$$\theta_u(x) := \int_{u^\wedge(x)}^{u^\vee(x)} \theta_{\{u>t\}}(x) \, dt.$$

Crucially, neither the absolutely continuous part $a \, d\mu$ nor the Cantor part "sees" sets of finite $\mathcal{H}$-measure.

Now, if Ω is a strong BV extension domain and $u \in \text{BV}(\Omega)$, the decomposition of the variation measure and the fact that $\|D(Eu)\|(\partial\Omega) = 0$ imply that $\mathcal{H}(\partial\Omega \cap S_{Eu}) = 0$. In other words, there is negligible intersection between the boundary of Ω and the "jump set" of Eu. This means that $\mathcal{H}$-almost every point of the boundary $\partial\Omega$ is a Lebesgue point of Eu [50, Theorem 3.5]. If Ω furthermore satisfies the weak measure density condition, it is straightforward to conclude that there is a boundary trace $T_\Omega u(x)$ for $\mathcal{H}$-almost every $x \in \partial\Omega$, satisfying

$$\lim_{r \to 0} \fint_{B(x,r) \cap \Omega} |u - T_\Omega u(x)| \, d\mu = 0.$$

Let us introduce one more definition. Based on [9, Definition 6.1], we say that X satisfies the *locality condition* if for any sets of finite perimeter E_1 and E_2, we have $\theta_{E_1}(x) = \theta_{E_2}(x)$ for $\mathcal{H}$-almost every $x \in \partial^* E_1 \cap \partial^* E_2$. Later we give examples of spaces that satisfy this condition. By using the locality condition and by establishing a few results on boundary traces, we are able to obtain formulation (8.1) when Ω is a strong BV extension domain and additionally satisfies a few geometric assumptions.

Which domains, then, are (strong) BV extension domains on metric spaces? In the Euclidean case it is a standard result that every bounded domain with a Lipschitz boundary is an extension domain for BV and Sobolev functions, see e.g. [7, Proposition 3.21]. For Sobolev functions, this result was extended in [43] to all so-called (ε, δ)-domains, and in [53] a result is given on BV extension domains in $\mathbb{R}^2$. By contrast, in the general metric setting the best known result seems to be one found in [12], where it is shown that, roughly speaking, if sets of finite perimeter can be extended from a domain, so can general BV functions. However, a simple geometric characterization or sufficient condition for BV extension domains seems to be missing.

On the other hand, results on the extension of Hajłasz-Sobolev and Newtonian functions on metric spaces have been given in [32] and [16]. On a metric space the concept of a domain with a Lipschitz boundary does not make sense, but a natural generalization is a *uniform domain*. For $A \geq 1$, a domain Ω is A-uniform if for every $x, y \in \Omega$ there is a curve γ connecting x and y in Ω such that $\ell_\gamma \leq A d(x,y)$ and

$$\text{dist}(\gamma(t), X \setminus \Omega) \geq A^{-1} \min\{t, \ell_\gamma - t\}$$

for every $t \in [0, \ell_\gamma]$, where ℓ_γ is the length of the curve — recall that we assume all curves to be parametrized by arc-length. Based on results concerning the extension of Newtonian functions, given by Björn and Shanmugalingam in [16], we show in Publication V that any bounded uniform domain is a strong BV extension domain.

It may be mentioned here that Whitney coverings and discrete convolutions, which were discussed in Section 4, are a central tool in the extension of various types of functions u from a set Ω (not always open) to the whole space. One example is the classical Whitney's extension theorem for smooth functions, see e.g. [22, p. 245]. Typically the idea is to take a Whitney covering $\{B(x_j, r_j)\}_{j=1}^{\infty}$ of the open set $X \setminus \overline{\Omega}$, and the corresponding partition of unity $\{\varphi_j\}_{j=1}^{\infty}$. For each x_j we pick a point $y_j \in \overline{\Omega}$ satisfying $d(x_j, y_j) = \mathrm{dist}(x_j, \overline{\Omega})$, and then define

$$Eu(x) := \sum_{j=1}^{\infty} u(y_j)\varphi_j(x), \qquad x \in X \setminus \overline{\Omega}.$$

Depending on the set Ω and the class of functions that we wish to extend, the number $u(y_j)$ may not be well defined, and then we may replace it with an integral average of u over $B(y_j, r_j) \cap \Omega$. In this way, we obtain a "discrete convolution" of u in $X \setminus \overline{\Omega}$, but not in the usual sense, since the function u is not defined in this set. Instead, the coefficients for the functions φ_j are obtained by "reflecting" values of u from $\overline{\Omega}$. In [16] this method is used to extend Newtonian functions, and in [15] it is used to extend characteristic and Hölder continuous functions. Of course, in the Euclidean case, extensions can be defined by actually reflecting (in the ordinary sense) a function across the boundary of a domain with respect to some coordinate axis, but this would not make sense on a metric space.

The approach to constructing boundary traces of BV functions which is used in Publication IV and which we have discussed so far is based on the fairly strong assumption that Ω is a strong BV extension domain. In Publication V we consider a different approach, in which more is assumed of the space X but less of the domain Ω. We say that a space X satisfies the *strong locality condition* if for any sets of finite perimeter $E_1 \subset E_2 \subset X$ and $\mathcal{H}$-almost every $x \in \partial^* E_1 \cap \partial^* E_2$, we have

$$\lim_{r \to 0} \frac{\mu(B(x, r) \cap E_2 \setminus E_1)}{\mu(B(x, r))} = 0. \tag{8.3}$$

As the name implies, this condition implies the (ordinary) locality condition discussed earlier in this section, by Lemma 4.5 in Publication V and [9, Proposition 6.2]. The strong locality condition is satisfied e.g. in Euclidean spaces, and also in any weighted Euclidean space where the weight is locally bounded and locally bounded away from zero, see Example 4.6 and Example 4.7 in Publication V. In Euclidean spaces, any set of finite perimeter converges, under blow-up, to a half-space at $\mathcal{H}$-almost every point of the set's measure theoretic boundary. In particular, sets of finite perimeter have density half in these points, guaranteeing the strong locality condition.

In a metric space, the concept of half-spaces does not make sense, but the strong locality condition seems to summarize the crucial information required for results on traces. Moreover, we are able to use condition (8.3) for some sets of finite perimeter even without assuming that the space satisfies the strong locality condition. Indeed, assume that $A \subset X$ consists of points $x \in \partial^* E_1 \cap \partial^* E_2$ for which (8.3) fails, i.e. the limit superior is

positive. Then we have $x \in \partial^*(E_2 \setminus E_1)$ for every $x \in A$, and we know that $E_2 \setminus E_1$ is also a set of finite perimeter. By (8.2) we then have

$$\liminf_{r \to 0} \frac{\mu(B(x,r) \cap E_2 \setminus E_1)}{\mu(B(x,r))} \geq \gamma > 0$$

for $\mathcal{H}$-almost every $x \in A$. From this it follows that if we have a sequence of sets of finite perimeter $E_1 \subset E_2 \subset \ldots \subset X$, and in a set $A \subset \partial^* E_1 \cap \partial^* E_2 \cap \ldots$, with $\mathcal{H}(A) > 0$, the condition (8.3) fails for all pairs E_{i+1}, E_i, $i = 1, 2, \ldots$, then the number of these sets is at most a constant depending only on the number γ. On the other hand, we know that for any point $x \in S_u$ in the jump set of a BV function, $x \in \partial^*\{u > t\}$ for all $t \in (u^\wedge(x), u^\vee(x))$. This follows from the definitions of the lower and upper approximate limits $u^\wedge(x)$ and $u^\vee(x)$. In Publication V we use this type of deductions to conclude that at $\mathcal{H}$-almost every point $x \in S_u$, there is a number $t_2 \in (u^\wedge(x), u^\vee(x))$ such that

$$\lim_{r \to 0} \frac{\mu(B(x,r) \cap \{u > t_2\} \setminus \{u > t\})}{\mu(B(x,r))} = 0$$

for all $t < u^\vee(x)$. Then, by using reasoning similar to that presented in [22, p. 214], we are able to conclude that at $\mathcal{H}$-almost every $x \in S_u$,

$$\lim_{r \to 0} \fint_{B(x,r) \cap \{u > t_2\}} |u - u^\vee(x)|^{Q/(Q-1)} \, d\mu = 0; \tag{8.4}$$

recall that the number $Q > 1$ was defined in (2.3). An analogous result holds for $u^\wedge(x)$, with the set $\{u > t_2\}$ replaced by $\{u \leq t_1\}$, for some $t_1 \in (u^\wedge(x), u^\vee(x))$. These results describe the behavior of a BV function in its jump set, strengthening [50, Theorem 1.1], where the limit of $u_{B(x,r)}$ as $r \to 0$ is studied at points $x \in S_u$.

In Publication V we also show that if we then assume that the space satisfies the strong locality condition, we can choose $t_1 = t_2 = t$ in the above results, and in fact t_1 and t_2 can be chosen freely from the interval $(u^\wedge(x), u^\vee(x))$. This strongly resembles the Euclidean result, according to which the sets $\{u \leq t\}$ and $\{u > t\}$ can be replaced by complementary half-spaces, see e.g. [22, p. 213]. Again, the concept of a half-space does not make sense in the metric setting, where the sets $\{u \leq t\}$ and $\{u > t\}$ may not even have density $1/2$ at x, but these level sets are nonetheless complementary sets, and their lower and upper densities are controlled by (8.2).

Returning to the concept of traces, in Publication V we use the strong locality condition to show the existence of *interior traces* on the measure theoretic boundary of any set of finite perimeter. More precisely, let $\Omega \subset X$ be a set of finite perimeter, and let $u \in \mathrm{BV}(X)$. Then for $\mathcal{H}$-almost every $x \in \partial^*\Omega$ there exist interior traces $\{T_\Omega u(x), T_{X \setminus \Omega} u(x)\} = \{u^\wedge(x), u^\vee(x)\}$, which satisfy

$$\lim_{r \to 0} \fint_{B(x,r) \cap \Omega} |u - T_\Omega u(x)|^{Q/(Q-1)} \, d\mu = 0$$

and

$$\lim_{r \to 0} \fint_{B(x,r) \setminus \Omega} |u - T_{X \setminus \Omega} u(x)|^{Q/(Q-1)} \, d\mu = 0.$$

The idea of the proof is the following. If a point $x \in \partial^*\Omega$ is not in the jump set S_u, then it is, excluding a $\mathcal{H}$-negligible set, a Lebesgue point of u. Then we can define both interior traces simply as the Lebesgue limit of u at x. On the other hand, assume that $x \in S_u$. Now the function u makes a "jump" at x from $u^\wedge(x)$ to $u^\vee(x)$. By the strong locality condition, we know that the level sets $\{u > t\}$, for $t \in (u^\wedge(x), u^\vee(x))$, look locally either like the set Ω or its complement. Thus, near the point x, the function u is close to $u^\vee(x)$ in Ω and close to $u^\wedge(x)$ in $X \setminus \Omega$, or vice versa, and by these facts we are eventually able to prove the trace results presented above.

Once we have interior traces, we can prove the existence of boundary traces on the measure theoretic boundary of certain sets of finite perimeter Ω. However, here we again need the additional assumption that Ω is a BV extension domain. With this assumption, we can simply extend any $u \in \mathrm{BV}(\Omega)$ to $Eu \in \mathrm{BV}(X)$, and then use the interior trace theorem for the function Eu. In this way we can again ultimately obtain the representation (8.1) for the functional $\mathcal{F}$.

In conclusion, the significant advantage of assuming the *strong* locality condition is that interior traces can be defined on the measure theoretic boundary of any set of finite perimeter. This is sometimes useful; for example, when proving the desired formulation (8.1) in Publication IV, we need the existence of the trace $T_{X\setminus\Omega}h$, where (in essence) $h \in \mathrm{BV}(X)$. One approach to ensuring that this trace exists is to assume that $X \setminus \overline{\Omega}$ is also a strong BV extension domain. However, if the space supports the strong locality condition, it is enough to assume that Ω is a set of finite perimeter with $\mathcal{H}(\partial\Omega \setminus \partial^*\Omega) = 0$. Moreover, the elegant description of the behavior of a BV function in its jump set, as given in (8.4), is essentially also an interior trace result.

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In this thesis we study functions of bounded variation, abbreviated as BV functions, on metric measure spaces. We always assume the space to be equipped with a doubling measure, and mostly we also assume it to support a Poincaré inequality.

A central topic in the thesis are the various characterizations of BV functions. We prove a pointwise characterization of BV functions, and we study the so-called Federer-type characterization of sets of finite perimeter.

Moreover, we study functionals of linear growth, which give a generalization of BV functions, and consider a related minimization problem. This also leads us to the study of boundary traces and extensions of BV functions.

BUSINESS +
ECONOMY

ART +
DESIGN +
ARCHITECTURE

SCIENCE +
TECHNOLOGY

CROSSOVER

DOCTORAL
DISSERTATIONS